Tests of General Relativity;
By Ranging to Satellites and the Moon

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$$G_{\mu\nu}(g_{\alpha\beta}, g_{\alpha\beta,\gamma}, g_{\alpha\beta,\gamma\eta}) = \kappa T_{\mu\nu}$$

General Relativity is a Field Theory of Gravity, with a 2nd Rank Tensor Field coupled to all other matter and fields in nature such that the background gravity tensor $g_{\alpha\beta}(x^{\mu})$ establishes a globally non-Euclidean, non-Minkowskian Riemannian arena for the metrology of rulers and clocks

The operational meaning of the coordinates used to label events emerges from the solutions of the full theory with source-matter; it is not a-priori.

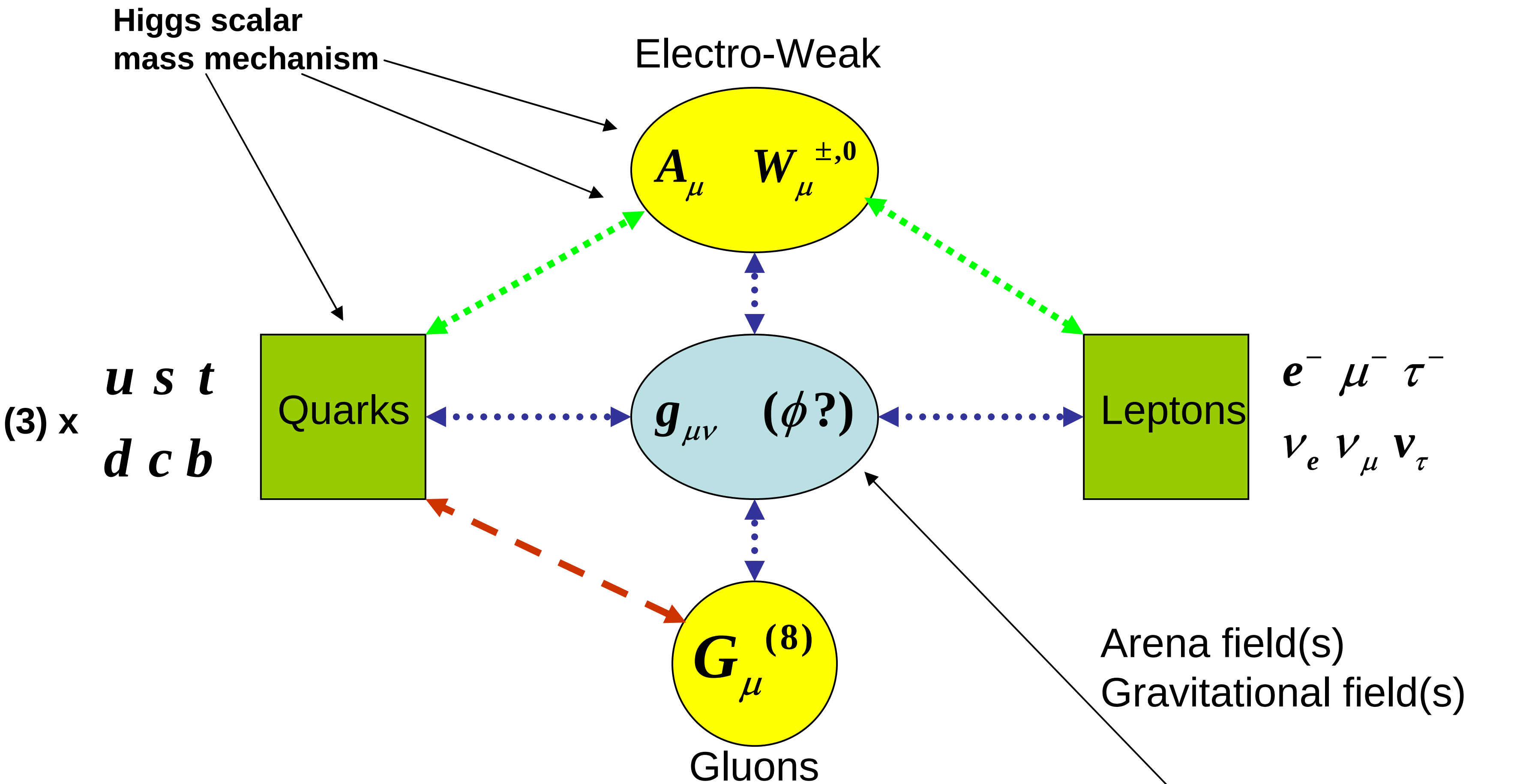
Light

$$g_{\mu\nu}(x^{\alpha})\, dx^{\mu} dx^{\nu} = 0$$

Clocks

$$d\tau = \sqrt{g_{\mu\nu}(x^{\alpha})\, dx^{\mu} dx^{\nu}}$$

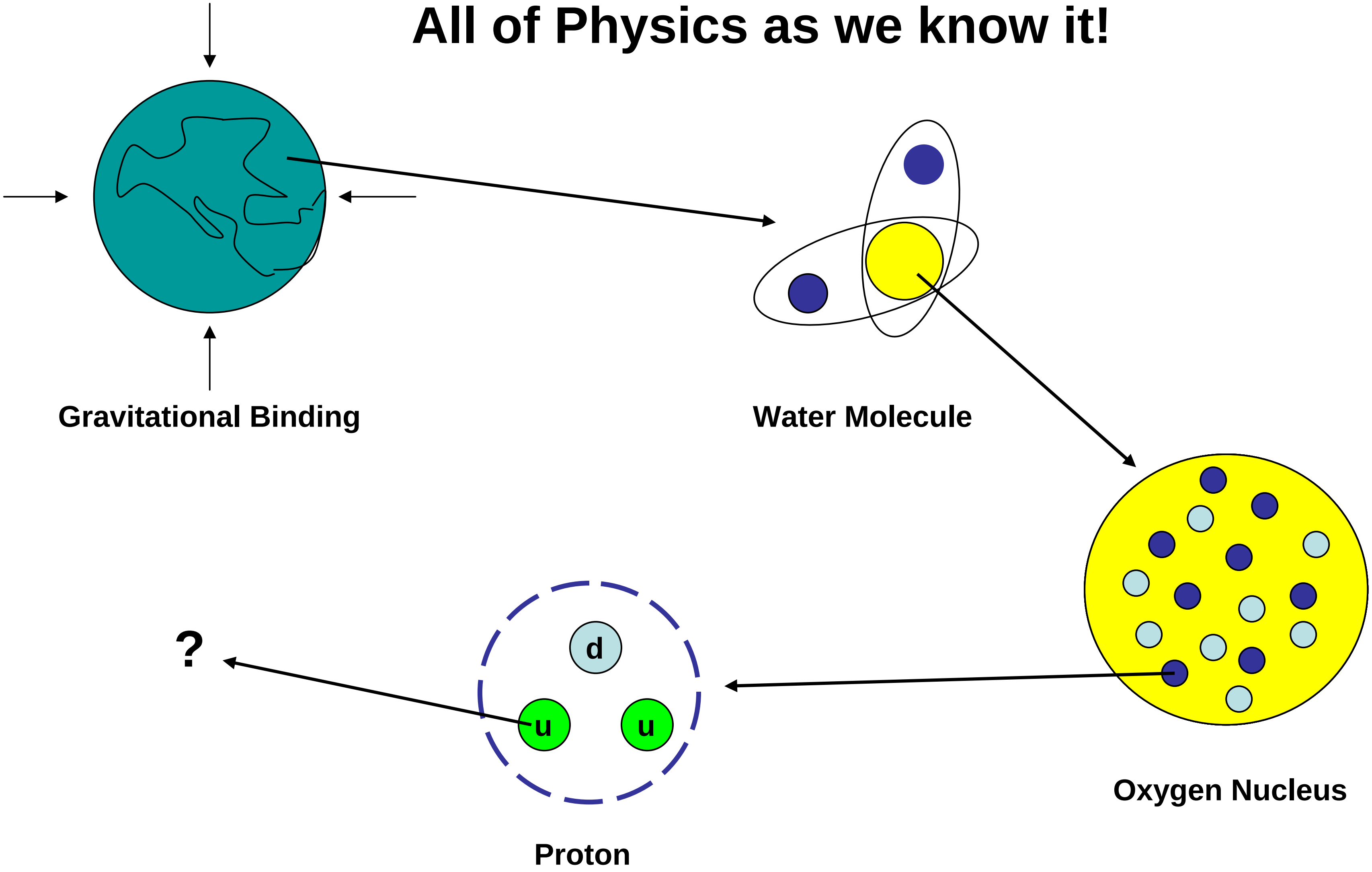
Standard Model plus Gravity



$$\text{"Rigid"} \quad \mathbf{1} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} g_{00} & \cdots & g_{03} \\ \vdots & \ddots & \vdots \\ g_{30} & \cdots & g_{33} \end{pmatrix}$$

$$g_{\mu\nu} = g(\vec{r}, t)_{\mu\nu} \quad \text{"Flexible"}$$

Earth's Mass-Energy consists of nuclear chromodynamic, electromagnetic, weak, kinetic, and gravity contributions. All of Physics as we know it!



Parameterized Post-Newtonian (PPN) Metric Field Expansion

For an Assembly of Mass Elements m_i

$$g_{oo} = (1-U)^2 + \beta^* \left(\frac{G^2}{c^4} \sum_{i,j} \frac{m_i m_j}{|\vec{r}-\vec{r}_i|} \left(\frac{1}{|\vec{r}-\vec{r}_j|} + \frac{2}{|\vec{r}_i-\vec{r}_j|} \right) \right) - (2\gamma+1) \frac{G}{c^4} \sum_i \frac{m_i v_i^2}{|\vec{r}-\vec{r}_i|} + \ldots$$

$$g_{ab} = -\delta_{ab} \left(1 + 2\gamma U\right) + \ldots$$

$$g_{oa} = (2\gamma+2) \frac{G}{c^3} \sum_i \frac{m_i (\vec{v})_a}{|\vec{r}-\vec{r}_i|} - \frac{G}{2c^3} (\vec{\nabla})_a \sum_i \frac{m_i \vec{v}_i \cdot (\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|} + \ldots$$

with $U = \frac{G}{c^2} \sum_i \frac{m_i}{|\vec{r}-\vec{r}_i|}$ **and** $\vec{r}(t)_i$ **being instantaneous positions**

But for Assembly of Celestial Bodies

$$g_{oo} = 1 - 2\frac{G}{c^2} \sum_i \frac{M(G)_i}{|\vec{r}-\vec{r}_i|} + \ldots \qquad g_{ab} = -\delta_{ab} \left(1 + 2\gamma \frac{G}{c^2} \sum_i \frac{M(\gamma)_i}{|\vec{r}-\vec{r}_i|} \right) + \ldots$$

with gravitational mass $M(G)$ and "gamma mass" $M(\gamma)$ generally differing from body inertial masses $M(I)$

$$M(G) = M(I) + (2\beta^* - \gamma - 1)\, V_G$$

$$M(\gamma) = M(I) + \chi^{2PN}\; V_G$$

with V_G being body's gravitational binding energy/c^2

V_G / Mc^2 for Sun is $4\,10^{-6}$

$$\left(\frac{M(G)}{M(I)}\right)_{Earth} = (2\beta^* - \gamma - 1)\quad 5\,10^{-10}$$

Post-Newtonian Equation of Motion for N Bodies

$$\vec{a}_i = \left(1 + \frac{dG}{Gdt}(t - t_o)\right)\left(\frac{m_G}{m_I}\right)_i \vec{g}_i \qquad \textbf{Modified Newtonian}$$

$$+\frac{1}{c^2}\left[\begin{array}{ll}
-\beta^* \sum_{j\neq i}\left(\sum_{k\neq i}\frac{\mu_k}{r_{ik}} + \sum_{k\neq j}\frac{\mu_k}{r_{jk}}\right)\vec{g}_{ij} & \textbf{Non-linearity} \\
+(2\gamma + 2)\,\vec{v}_i \times \left(\sum_{j\neq i}\vec{v}_j \times \vec{g}_{ij}\right) & \textbf{Gravitomagnetism} \\
+(\gamma + 1/2)\sum_{j\neq i}\left(v_i^{\,2}\vec{g}_{ij} - 2\vec{v}_{ij}\cdot\vec{g}_{ij}\,\vec{v}_i\right) & \textbf{Geodetic Precession} \\
+\sum_{j\neq i}\left(\left[(\gamma + 1)v_j^{\,2} - 3(\vec{v}_j\cdot\hat{r}_{ij})^2/2\right]\vec{g}_{ij} - (2\gamma + 1)\vec{v}_j\vec{v}_j\cdot\vec{g}_{ij}\right) & \textbf{Source Motion} \\
+\sum_{j\neq i}\frac{\mu_j}{2r_{ij}}\vec{a}_j - \frac{1}{2}v_i^{\,2}\vec{a}_i & \textbf{Inertial} \\
-(2\gamma + 1)\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\left(\vec{a}_i - \vec{a}_j\right) + \left(\sum_{j\neq i}\frac{\mu_j}{2r_{ij}}\vec{a}_j\cdot\hat{r}_{ij}\,\hat{r}_{ij} - \vec{a}_i\cdot\vec{v}_i\,\vec{v}_i\right) & \textbf{Misc. Inertial}
\end{array}\right]$$

with $\mu_i = Gm_i$, $\quad \vec{g}_{ij} = \frac{Gm_j}{r_{ij}^{\,3}}\vec{r}_{ji}$, $\quad \vec{g}_i = \sum_{j\neq i}\vec{g}_{ij}$ **and** $\gamma = 1$, $\beta^* = 1$ **in GR**

The Post-Newtonian Lagrangian for N masses

$$L = -\sum_i m_i \left(c^2 - \frac{u_i^2}{2} - \frac{u_i^4}{8c^2} \right) + \frac{G}{2} \sum_{i \neq j} \frac{m_i m_j}{r_{ij}} \left(1 - \frac{\vec{u}_i \cdot \vec{u}_j + \vec{u}_i \cdot \hat{r}_{ij} \hat{r}_{ij} \cdot \vec{u}_j}{2c^2} \right)$$

$$+ \frac{1 + 2\gamma}{4} \frac{G}{c^2} \sum_{i \neq j} \frac{m_i m_j}{r_{ij}} \left(\vec{u}_i - \vec{u}_j \right)^2 - \frac{\beta^*}{2} \frac{G^2}{c^2} \sum_{i \neq j,k} \frac{m_i m_j m_k}{r_{ij} r_{ik}}$$

$\gamma = \beta^* = 1$ **in General Relativity**

$$\vec{P} = \sum_i \vec{p}_i = \sum_i \frac{\partial L}{\partial \vec{u}_i} \qquad E = \sum_i \vec{p}_i \cdot \vec{u}_i - L \qquad \vec{J} = \sum_i \vec{r}_i \times \vec{p}_i$$

$$\vec{R}_{CE} = \frac{1}{E} \sum_i m_i \vec{r}_i \left(c^2 + \frac{1}{2} u_i^2 - \frac{G}{2} \sum_{j \neq i} \frac{m_j}{r_{ij}} \right) \qquad \vec{V}_{CE} = \frac{c^2 \vec{P}}{E} \qquad \frac{d^2 \vec{R}_{CE}}{dt^2} = 0$$

$$\frac{d\vec{p}_i}{dt} = \frac{\partial L}{\partial \vec{r}_i} \quad \text{for each body } i \qquad \frac{d\vec{P}}{dt} = \frac{dE}{dt} = \frac{d\vec{J}}{dt} = 0$$

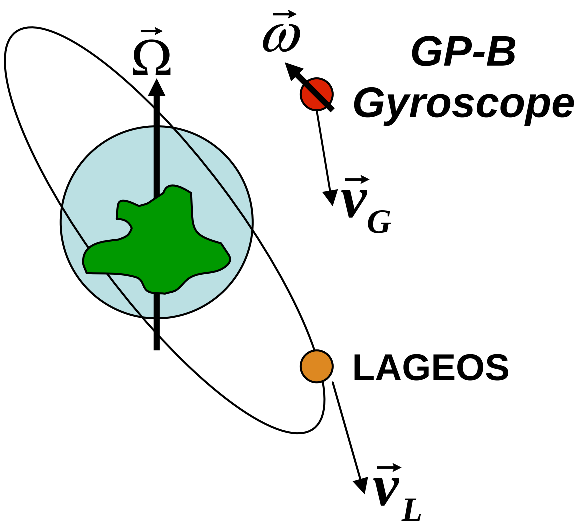
"Gravitomagnetism" --- A Relativistic Gravity Acceleration

Between Two Mass Currents

$$\vec{a}_i = 2(1+\gamma)\,\vec{v}_i \times \sum_j \frac{Gm_j}{c^2 r_{ij}^3}\left(\vec{v}_j \times \vec{r}_{ij}\right) \quad - \quad \frac{1}{2}\vec{v}_i \times \vec{\nabla}_i \sum_j \frac{Gm_j}{c^2 r_{ij}} \vec{v}_j \cdot \vec{r}_{ij}$$

A Few Manifestations

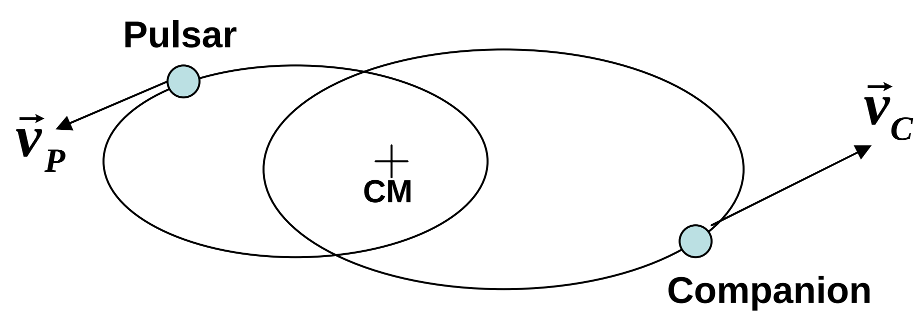
Orbitting Probes Near Spinning Earth



Binary Pulsar System

$$\delta\vec{a} = 2(1+\gamma)\frac{Gm_P m_C}{Mc^2 r^3}\left(u^2\vec{r} - \vec{u}\cdot\vec{r}\,\vec{u}\right)$$

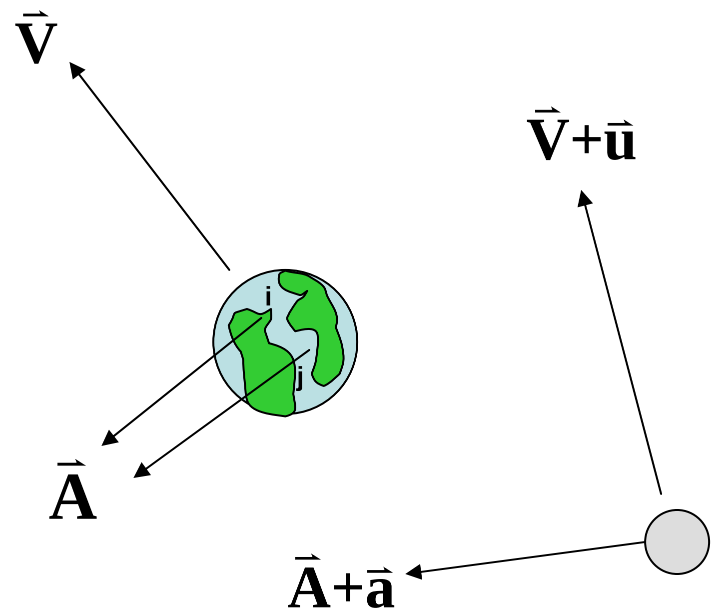
$$\vec{r} = \vec{r}_P - \vec{r}_C \qquad \vec{u} = \vec{v}_P - \vec{v}_C$$



Sun-Earth-Moon System is Comprehensive Relativistic Test Bed

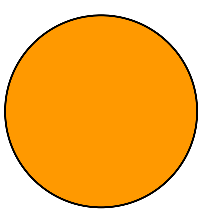
Gravitomagnetism: Earth and Moon exert gravitational forces on each other in proportion to both their motions, $\vec{V}$ and $\vec{V}+\vec{u}$.

Geodetic Precession: Sun's acceleration of Moon differs from Sun's acceleration of Earth because of their different velocities $\vec{V}+\vec{u}$ and $\vec{V}$.



Earth's Gravitational Mass: The non-linear gravity field of (Sun + "i") exerts force on Earth's element "j".

Earth's Inertial Mass: Acceleration of mass element "i" induces an acceleration of Earth's mass element "j".



If the gravito-magnetic forces between moving Earth and moving Moon were turned off, there would be anomalous 500 centimeter amplitude oscillations in Earth-Moon range of two periods --- monthly and half sidereal month.

Three secular LAGEOS orbital perturbations,

Nodal Precession Rates, ω_1 and ω_2 plus Perigee Precession Rate ω_2^*

are driven by Gravitomagnetism and Earth Gravity Multipoles

$$\omega_1 = K_{11}\ \lambda_{LT} + K_{12}\ J_2 + K_{13}\ J_4 + \ldots$$

$$\omega_2 = K_{21}\ \lambda_{LT} + K_{22}\ J_2 + K_{23}\ J_4 + \ldots$$

$$\omega_2^* = K_{31}\ \lambda_{LT} + K_{32}\ J_2 + K_{33}\ J_4 + \ldots$$

with coefficients $K_{ij}(\theta)$ calculable for each LAGEOS orbit.

Gravitomagnetic parameter λ_{LT} can be solved for in face of a-priori uncertainties in Earth's J_2 and J_4.

$$\lambda_{LT} = C_1\omega_1 + C_2\omega_2 + C_2^*\omega_2^*$$

I. Ciufolini, *G.R.G.* 36 (10), 2004

Gravity-Induced Doppler Signals From Cassini During Line-of-Sight Passage By The Sun

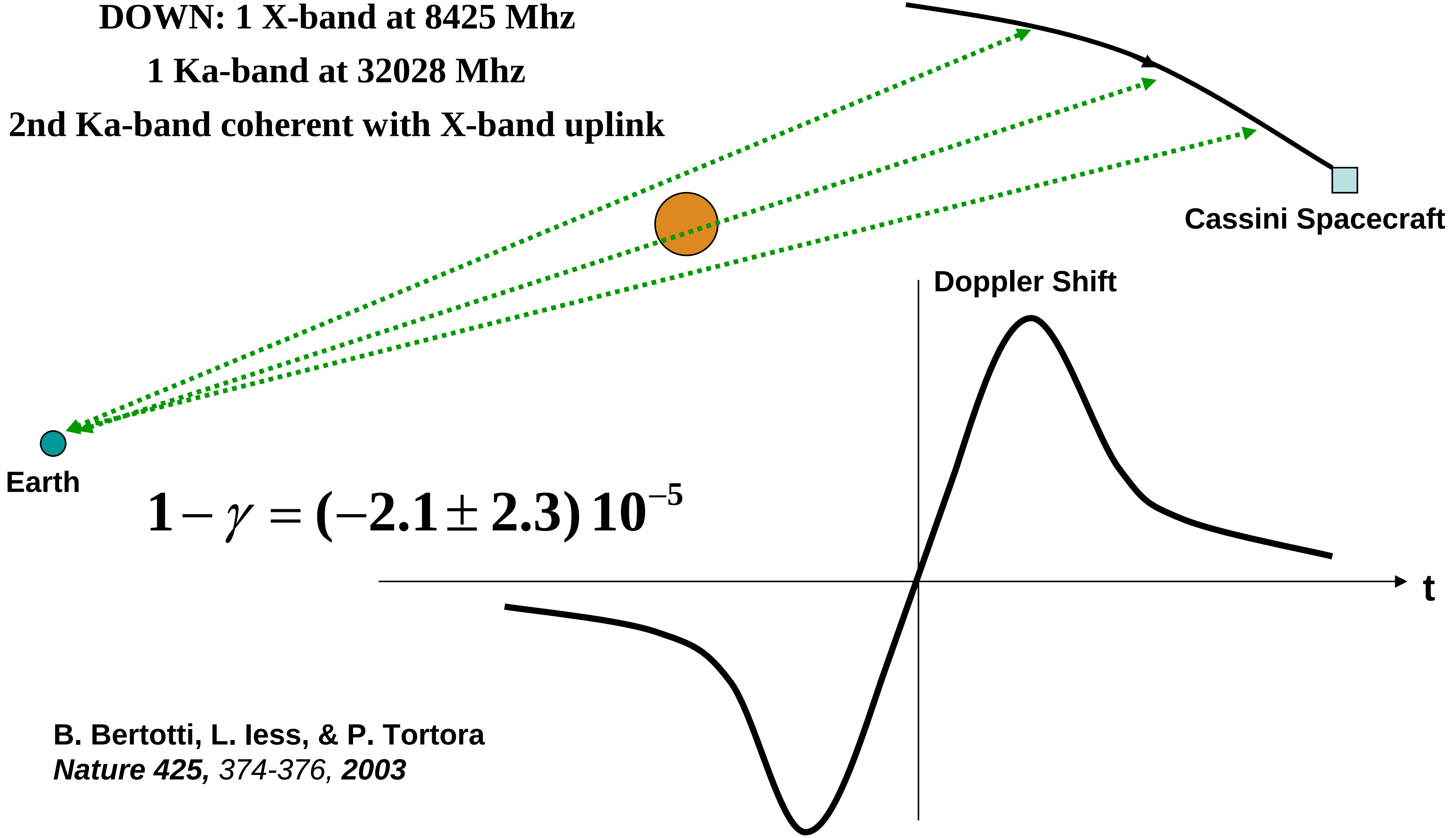
UP: 2 X-band at 7175 Mhz

1 Ka-band at 34316 Mhz

DOWN: 1 X-band at 8425 Mhz

1 Ka-band at 32028 Mhz

2nd Ka-band coherent with X-band uplink



$$1-\gamma = (-2.1 \pm 2.3)\,10^{-5}$$

B. Bertotti, L. Iess, & P. Tortora
Nature 425, *374-376,* ***2003***

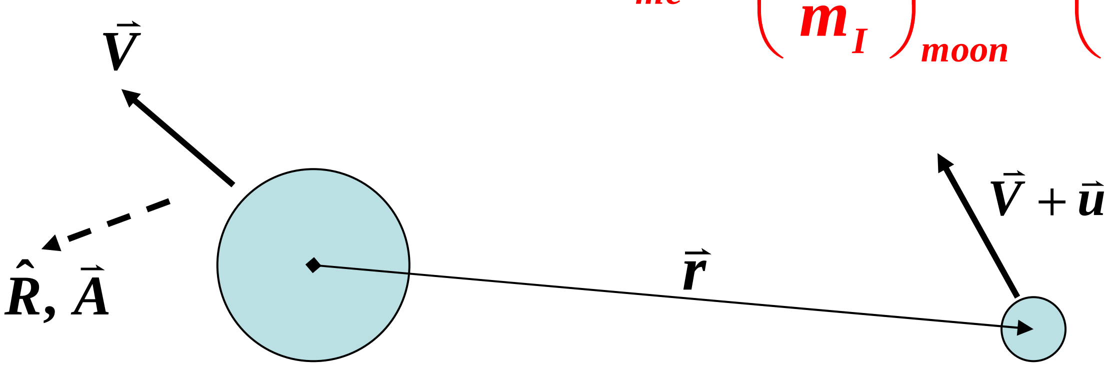
Structure of Moon's Equation of Motion Relative to Earth

$$\frac{d^2\vec{r}}{dt^2} = -\frac{Gm_e}{r^3}\vec{r}\left(1 + \frac{\dot{G}}{G}(t - t_o)\right)$$

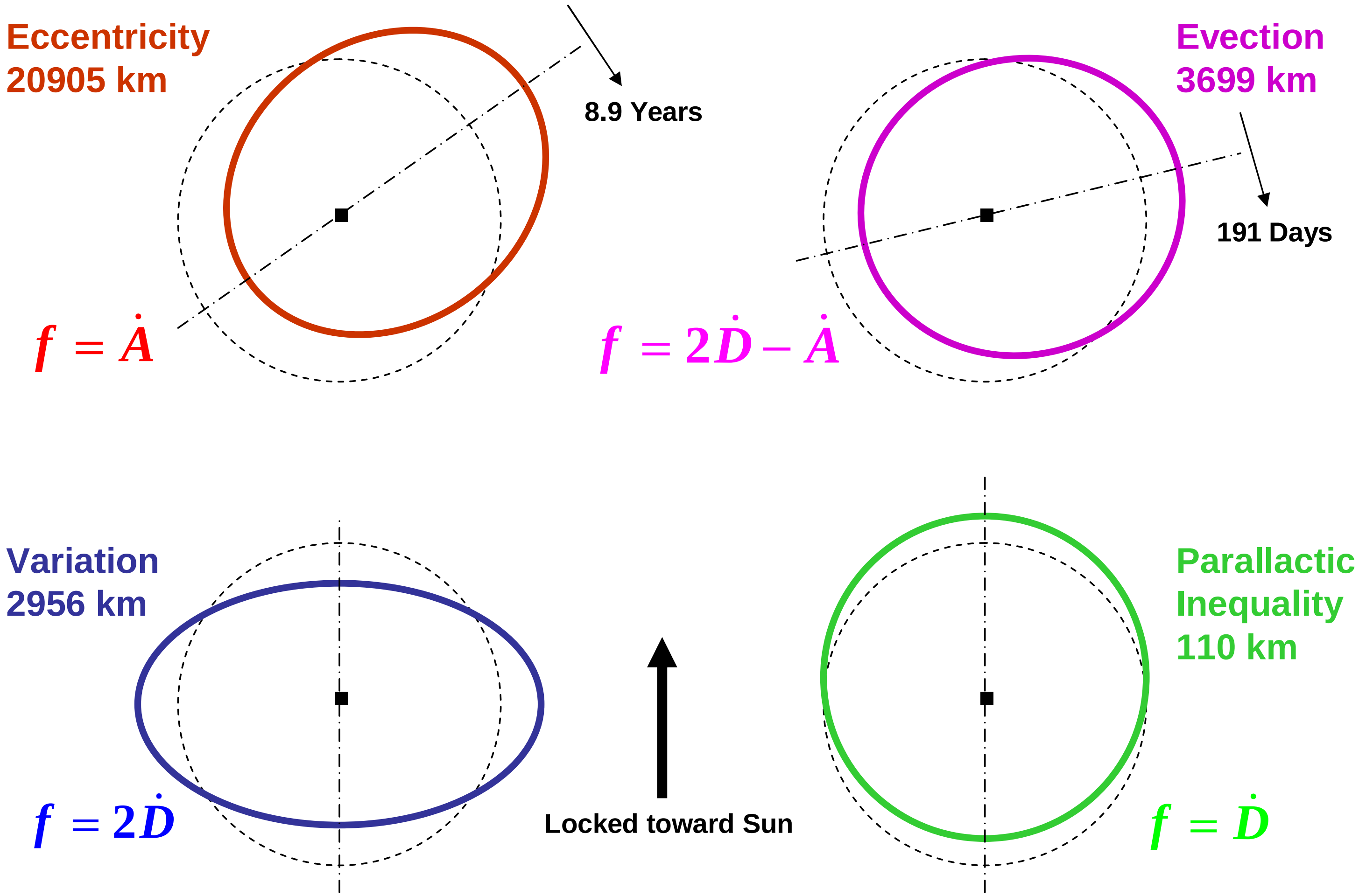
$$+\frac{GM_s}{R^3}\left(\delta_{me}\vec{R} + \left[3\hat{R}\hat{R}\cdot\vec{r} - \vec{r}\right] - \frac{1}{2R}\left[6\vec{r}\cdot\hat{R}\,\vec{r} + 3r^2\hat{R} - 15\left(\vec{r}\cdot\hat{R}\right)^2\hat{R}\right] + ...\right)$$

$$+\frac{1}{c^2}\left(\vec{g}_s(\vec{R},\vec{V}+\vec{u}) - \vec{g}_s(\vec{R},\vec{V})\right) + \frac{1}{c^2}\vec{g}_e(\vec{r},\vec{V}+\vec{u},\vec{V}) + ...$$

with $\delta_{me} = \left(\frac{m_G}{m_I}\right)_{moon} - \left(\frac{m_G}{m_I}\right)_{earth}$



Four Key Lunar Orbit Perturbations or "Inequalities"



K. Nordtvedt, LLR, 25 July 2003

$$R(t) = R_o - R_{ecc}\cos(A) - R_{evc}\cos(2D - A) - R_{var}\cos(2D) - R_{pi}\cos(D) + ...$$

Anomalistic phase (perigee to perigee) $A = A_o + \dot{A}(t - t_o) + \frac{1}{2}\ddot{A}(t - t_o)^2$

Synodic phase (new moon to new moon) $D = D_o + \dot{D}(t - t_o) + \frac{1}{2}\ddot{D}(t - t_o)^2$

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1. Measurement of R_{pi} tests whether Sun identically accelerates Earth and Moon.

$$\delta R_{pi} \simeq \frac{3}{2}\frac{V_e}{\dot{L}}\Re(\Omega/\omega)\frac{|\vec{a}_e - \vec{a}_m|}{|\vec{g}_s|} \simeq 3\,10^{12}\frac{|\vec{a}_e - \vec{a}_m|}{|\vec{g}_s|}\ cm.$$

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2. Measurement of phase accelerations $\ddot{A}$ and $\ddot{D}$ tests whether G varies in time.

$$\dot{D} = \dot{L} - \dot{L}_S \qquad \dot{A} = \dot{L}\left(1 - \frac{3}{4}\frac{\Omega^2}{\omega^2} - \frac{225}{32}\frac{\Omega^3}{\omega^3} - ...\right)$$

$\dot{L}$ is Moon's sidereal rate $\qquad \dot{L}_S$ is Earth's sidereal rate around Sun

$$\frac{\ddot{L}_S}{\dot{L}_S} = \ddot{A} - \ddot{D} + ... \quad \text{and} \quad \frac{\dot{G}}{G} = \frac{1}{2}\frac{\ddot{L}_S}{\dot{L}_S}$$ independent of tide-induced $\ddot{L}$

Equivalence Principle Violating Signal

$$\delta r_{em} \simeq \left(1 + 2\frac{\dot{L}}{\dot{D}}\right)\frac{\delta a_{em}}{\dot{L}^2 - \dot{D}^2}\cos D \simeq \frac{3}{2\dot{L}\,\dot{L}_S}\delta a_{em}\;\;\cos D$$

Including Feedback from Variation

$$\delta r_{em} \simeq \frac{3}{2\dot{L}\dot{L}_S}\left(\frac{1 - 4\dot{L}_S/\dot{L} + ...}{1 - 7\dot{L}_S/\dot{L} + ...}\right)\delta a_{em}\;\;\cos D$$

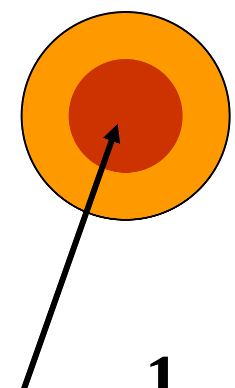
$$\delta r_{em} \simeq 3\,10^{12}\,|\,\delta a_{em}/g_S\,|\;\;\cos D\;\;\; cm$$

$$\delta r_{em} \simeq 0 \pm 4\,mm \qquad |\,\delta a_{em}/g_S\,| \simeq 0\;\; \pm 1.3\,10^{-13}$$

$$|\,\delta a_{Fe-Si}\,| \leq 4\,10^{-13}$$

$$(2\beta^* - 1 - \gamma)\frac{1}{M_e c^2}\sum_{i,j}\frac{Gm_i m_j}{2r_{ij}} \leq 1.3\,10^{-13}$$

$$|\,\beta^* - 1\,| \leq 1.5\,10^{-4}$$



$$M_{core} \simeq \frac{1}{3}M_e$$

LLR's Results from Testing

Gravity's Contributions to Gravitational and Inertial Masses

$$4.45\,10^{-10}\,\eta \quad + \quad \frac{1}{3}\Delta_{Core-Mantle} \quad \leq \quad 1.3\,10^{-13}$$

$$\eta = 4\beta - 3 - \gamma \;\equiv\; 2\beta^{*} - 1 - \gamma$$

$\Delta_{Core-Mantle} \quad \leq \quad 5\,10^{-13}$ **from Lab experiments**

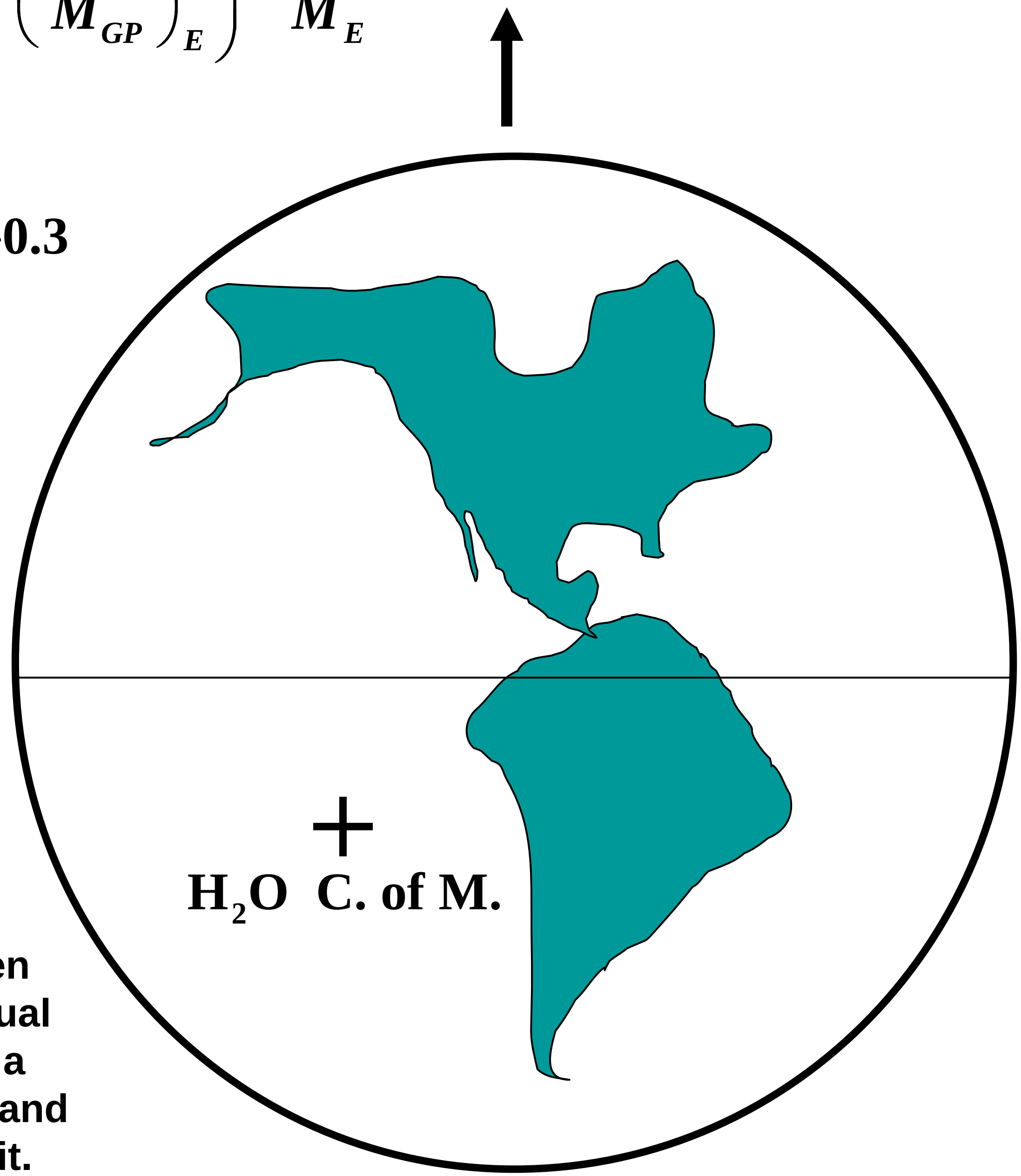
$|\,\eta\,| \leq 3\,10^{-4}$ **unless there's a chemical composition EP violation, also.**

$|1\text{-}\beta\,| \;\leq\; 10^{-4}$ **accepting Cassini's $|1\text{-}\gamma|$ measurement**

$$a = C_{ISO} \sin Lat \left(\left(\frac{M_{GA}}{M_{GP}}\right)_{H_2O} - \left(\frac{M_{GA}}{M_{GP}}\right)_E \right) \frac{M_{H_2O}}{M_E} g$$

$$\frac{M_{H_2O}}{M_E} \simeq 2.3\,10^{-4} \qquad \sin Lat \simeq -0.3$$

$$\left| \left(\frac{M_{GA}}{M_{GP}}\right)_{H_2O} - \left(\frac{M_{GA}}{M_{GP}}\right)_E \right| \leq 3\,10^{-13}$$



If the gravitational forces between Earth's water and core are not equal and opposite, Earth experiences a self-acceleration along polar axis, and consequently perturbs lunar orbit.

If any of the parameters of physical law vary in space-time, there generally will be forces acting on bodies.

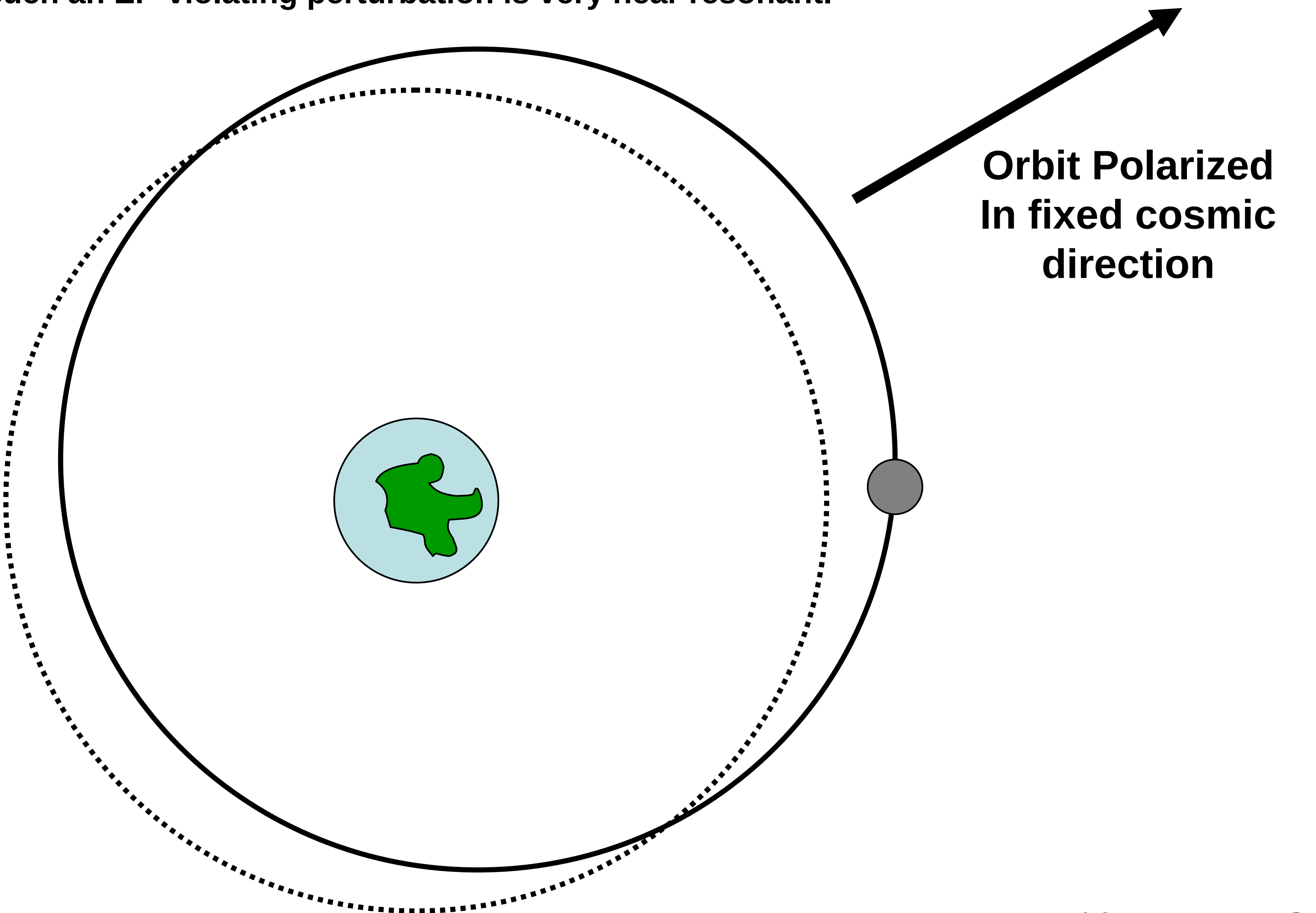
$$\vec{F} = -\ c^2 \frac{\partial M(P)}{\partial \ln P}\ \vec{\nabla} \ln P \quad \text{for} \quad P = G,\ \frac{e^2}{\hbar c},\ \ldots.$$

And any pure time dependence $P(t)$ only holds in preferred frame:

$$P(t) \rightarrow P\left(\frac{t' + \vec{v} \cdot \vec{r}\,'/c^2}{\sqrt{1 - v^2/c^2}}\right) \ \text{so}\ \vec{\nabla}' P = \frac{\partial P}{\partial t} \frac{1}{\sqrt{1 - v^2/c^2}} \frac{\vec{v}}{c^2}$$

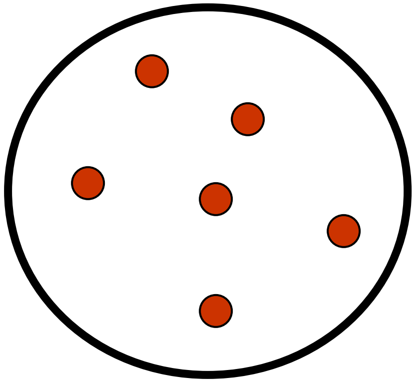
Cosmic or Sidereal EP Violation Test

Because lunar orbit's perigee precession period is 8.9 years, such an EP-violating perturbation is very near resonant.



$$|\vec{a}_e - \vec{a}_m|_{\text{cosmic}} \leq 1.2 \ \ 10^{-14} \ \ cm/s^2$$

LLR produces some interesting constraints on space-time variation of the fine structure 'constant'.



$$\delta \vec{a}_i = -\frac{1}{M_i}\frac{\partial M_i(\alpha)}{\partial \ln \alpha} c^2 \frac{\vec{\nabla}\alpha}{\alpha} \qquad \frac{\partial \ln M_i(Z)}{\partial \ln \alpha} \simeq 7.5\, 10^{-4} \frac{Z(Z-1)}{A^{4/3}}$$

$$c^2 \frac{|\vec{\nabla} \ln \alpha|}{|\vec{g}_S|} \le 10^{-10} \qquad |\vec{\nabla} \ln \alpha|_{Cosmic} \le 5\, 10^{-33}\ cm^{-1}$$

$$\frac{\delta\alpha}{\alpha} \le 5\, 10^{-5} \text{ across Universe} \qquad \frac{\delta\alpha}{\alpha} \le 1.5\, 10^{-10} \text{ across Galaxy}$$

$$\frac{|\vec{a}_e - \vec{a}_m|}{|\vec{g}_{gal}|} \le 5\, 10^{-8} \quad \text{Is dark matter strong source of EPV ?}$$

$$\frac{|\vec{a}_e - \vec{a}_m|}{10^{-5}(c^2 / R_{un})} \le 5\, 10^{-3} \quad \text{Is dark energy strong source of EPV?}$$