black holes

the basics: part I

The basic concepts and properties of black holes in general relativity

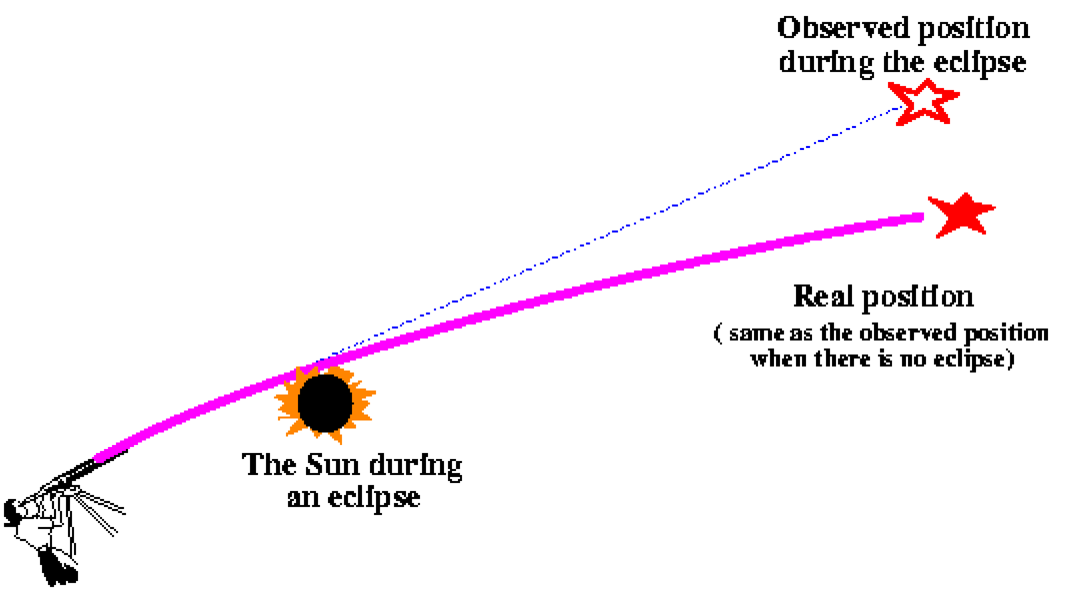
For the duration of this talk

$$\hbar = 0$$

Heuristic idea: "object" with gravity so strong that light cannot escape

Key concepts from general relativity that drive this idea:

1. Bending of light by gravity

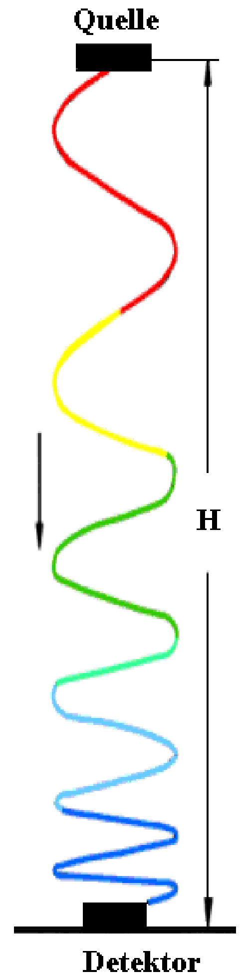


Heuristic idea: "object" with gravity so strong that light cannot escape

Key concepts from general relativity that drive this idea:

1. Bending of light by gravity

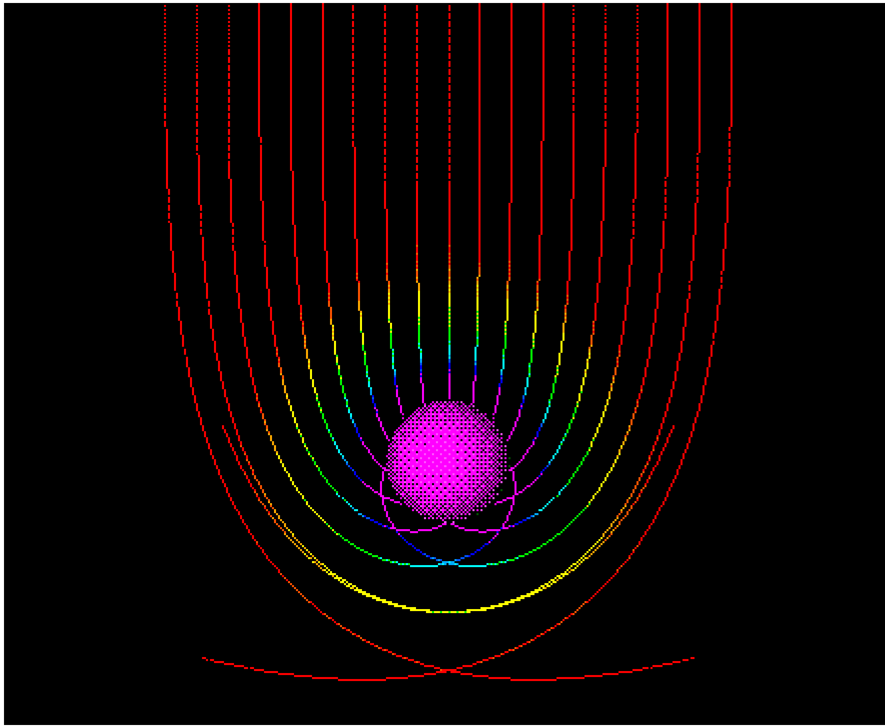
2. Gravitational redshift



Black holes: Concepts run amok!

Light bending so fierce that "bent" trajectories close …

Gravitational redshift powerful enough to *completely* "drain" photon energy!



Astrophysical context

Various observations reveal objects so massive and compact that the "black hole" hypothesis is *very* likely.

Several distinct mass bands are observed:

Stellar: ~2 Msun < M < 20 Msun. Appear to be the remnants of massive stars.

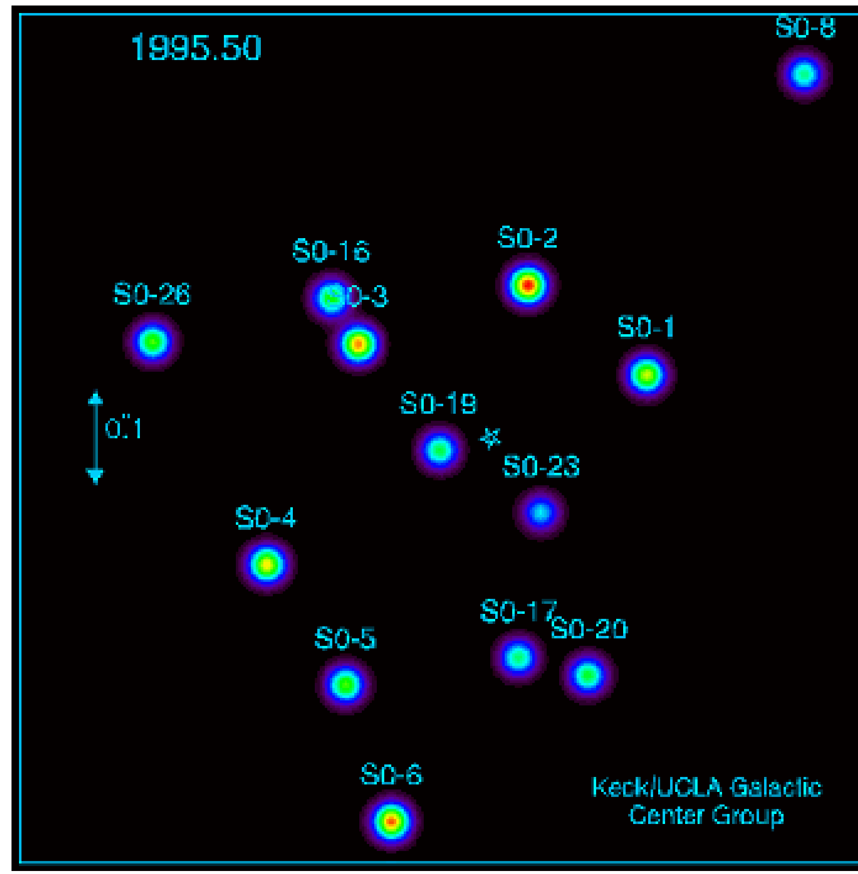
Massive/supermassive: ~a few 10^5 Msun < M < 10^9 Msun. Unclear how these originate!

"Intermediate" in between? Evidence accumulating, though not rock solid yet.

Astrophysical context

Various observations reveal objects so massive and compact that the "black hole" hypothesis is *very* likely.

Image courtesy Andrea Ghez

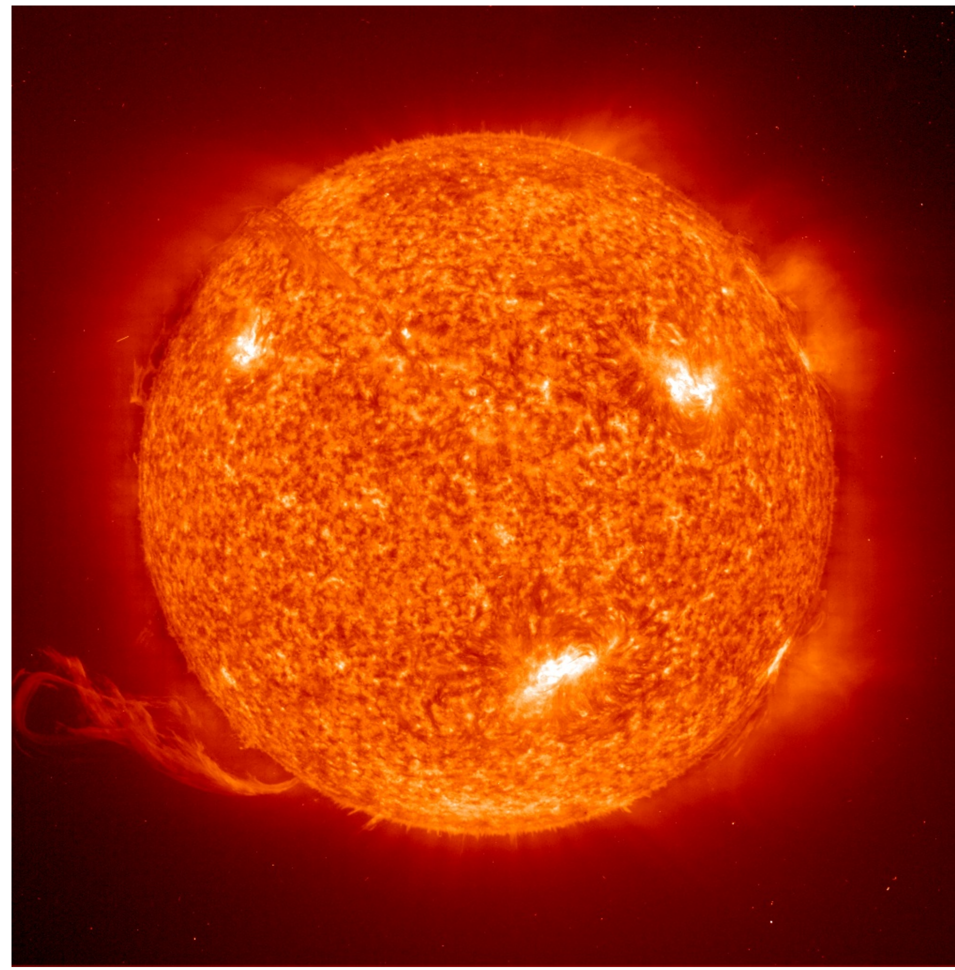


Are they black holes?

Need measurements that probe deep into the object's strong field, where we expect the strongest deviations from Newton's gravity.

Physical context

Black holes are astounding objects because of their *simplicity* – not unlike macroscopic elementary particles!



Consider a normal star: Lots of complicated physics set its macroscopic properties.

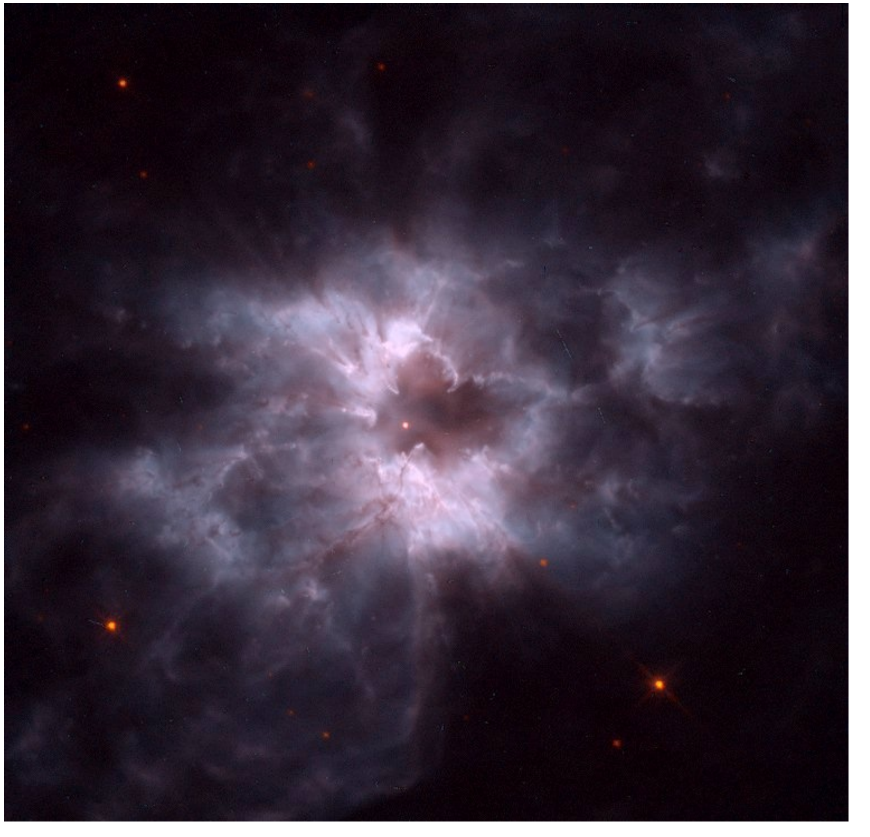
What happens when it dies?

Answer depends on the star's mass

Star exhausts its nuclear fuel – can no longer provide pressure to support itself. Gravity takes over, it collapses.

Low mass stars (< 8 Msun or so): End up with *white dwarf*, supported by electron degeneracy pressure.

Wide range of structure and composition!

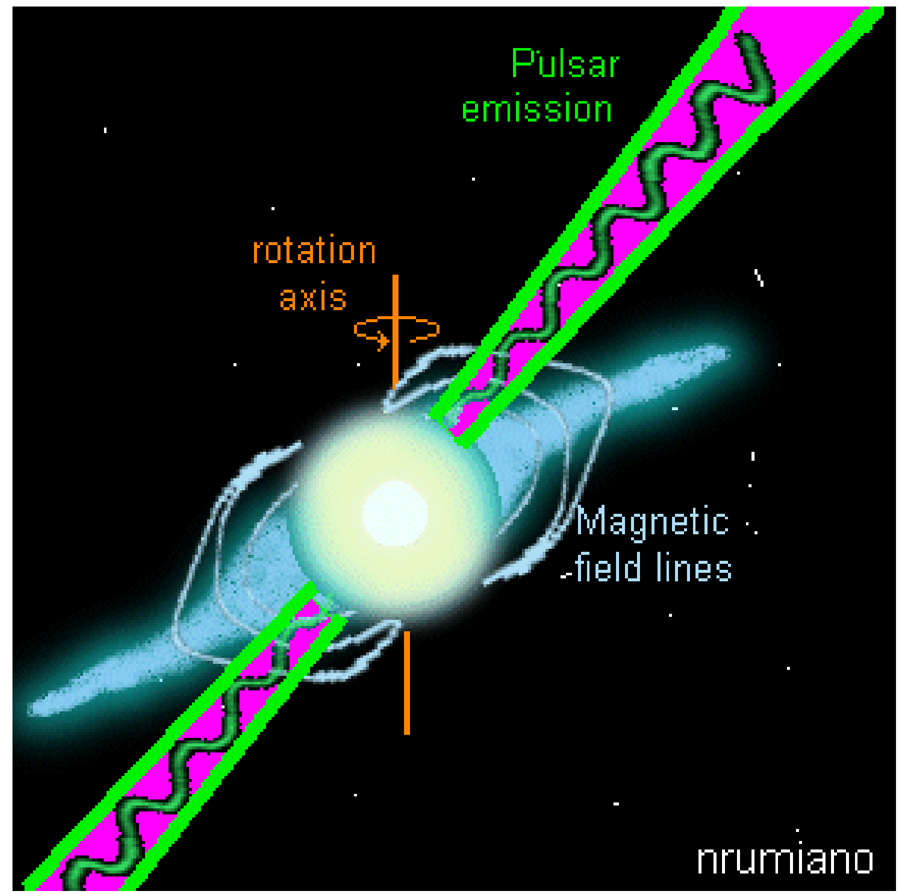


Answer depends on the star's mass

Star exhausts its nuclear fuel – can no longer provide pressure to support itself. Gravity takes over, it collapses.

Higher mass (roughly 8 – 25 Msun): Get a *neutron star*, supported by neutron degeneracy pressure.

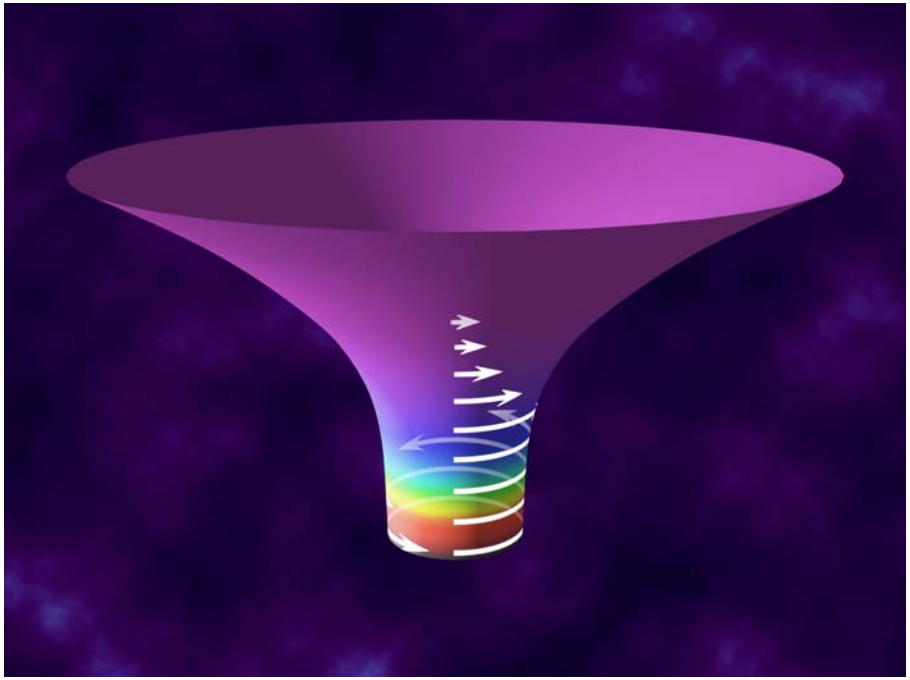
Wide variety of observed properties.



Answer depends on the star's mass

Star exhausts its nuclear fuel – can no longer provide pressure to support itself. Gravity takes over, it collapses.

Above about 25 Msun, we are left with a *black hole*.



According to general relativity, the black hole's properties are set by **two numbers:** *mass* and *spin*.

Goal for these lectures:

Why do we believe that black holes are inevitably created by gravitational collapse?

Rather astounding that we go from such complicated initial conditions to such "clean", simple objects!

Relativity units: $G = c = 1$

With these units, mass, length, and time all have the same dimension.

Important physical content for BH studies: general relativity has no intrinsic scale.

Instead, all important lengthscales and timescales are set by – and become proportional to – the mass.

Basic properties in GR

Simplest example: Schwarzschild solution, represents a non-rotating BH:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

Meaning: take weak field limit

$$ds^2 \simeq -(1 - 2M/r)\, dt^2 + (1 + 2M/r)\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

and examine "slow" ($v << c$) geodesics.

Result: These geodesics duplicate Newtonian gravity with source of mass M.

Interpretation: The Schwarzschild metric represents the "gravitational field" of a mass M in general relativity.

Basic properties in GR

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Some odd features immediately apparent:

- $r = 2M$ – apparent divergences.
  - Coordinate singularity, corresponding to "event horizon": Light cannot escape from this radius. Metric pathology can be eliminated by changing coordinates (though physical meaning is not).
- $r = 0$ – **more** divergences.
  - **Physical** singularity: *cannot* be removed.

Party trick:
Newtonian escape velocity

Calculation by Reverend John Michell

[Philosophical Transactions of the Royal Society of London **74**, 35-17 (1784)]

"Popularized" by Laplace

[*Exposition du Systeme du Monde*, 1796]

Trick: Abuse escape velocity formula.

$$\frac{1}{2} v_{esc}^2 = \frac{GM}{R}$$

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At what value of *R* does $v_{\rm esc} = c$?

Party trick:
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Trick: Abuse escape velocity formula.

$$R = R_{crit} = \frac{2GM}{c^2}$$

See also Matt Visser, "Heuristic approach to the Schwarzschild geometry", gr-qc/0309072

Schwarzschild metric: Details

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Easy to verify by direct substitution this is an exact solution of the vacuum ($T_{ab} = 0$) Einstein equations.

BUT: Is it a ***meaningful*** solution?

In other words, are the features of this mathematical solution features that are likely to be shared by objects in the real world?

Interpretation of coordinates

Begin by carefully studying the coordinates t, r, θ, and ϕ. Consider a "slice" of constant r and t; line element reduces to

$$ds^2 = r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

This is the metric of the surface of a sphere.

Means that Schwarzschild is a spherical solution, with θ and ϕ acting as "normal" spherical coordinates.

Radial coordinate r is ***not*** so simple!

Interpretation of coordinates

Consider now a "slice" in which *r* is allowed to vary:

$$ds^2 = \frac{dr^2}{1 - 2M/r} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

The coordinate *r* is an *areal* coordinate: Surfaces of constant *r* have the "normal" spherical surface area $A = 4\pi r^2$.

Distance between coordinates r_1 and r_2 at constant angle is ***not*** $\Delta s = \Delta r = r_2 - r_1$: Have to integrate the metric!

Interpretation of coordinates

Finally, what does the time coordinate *t* mean? Consider the limit *r* >>>> *M*:

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

This is the metric of flat spacetime!

Tells us that Schwarzschild is "asymptotically flat"; and, it tells us that the coordinate *t* corresponds to time ***as measured by distant observers***.

Summary of coordinates

Schwarzschild metric in Schwarzschild coordinates:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

- Spherically symmetric.
- Coordinate *r* labels surfaces of constant area, $A = 4\pi r^2$.
- Coordinate *r* does ***not*** label distances in a simple way! Must integrate.
- Coordinate *t* corresponds to time as measured by distant ($r >> M$) observers.

Birkhoff's Theorem

For further insight into Schwarzschild, consider the *general* static spherically symmetric line element:

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Reparameterize: $\Lambda(r) = -\frac{1}{2}\ln\left(1 - 2M/r\right)$

At this point, this is just a ***definition*** of *M*.

Next, require that this metric satisfy the vacuum Einstein equation $G_{ab} = 0$.

Doing so forces $\Phi(r) = -\Lambda(r)$ ... this leaves us with the Schwarzschild solution!

Meaning of Birkhoff's Theorem

This theorem tells us that the vacuum region outside *any* spherically symmetric gravitating source is given by the Schwarzschild metric.

In other words,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

describes the spacetime in the exterior of *any* spherical symmetric body of mass *M*.

Hence, *at the very least* the Schwarzschild metric is physically relevant for $r > 2M$. (True in all coordinate systems.)

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[Side note: This theorem holds even if the metric is *time varying*, as long as the variations are spherically symmetric (though details are more complicated).]

The event horizon

Big question: What is up with that singular-ish behavior at $r = 2M$?

Best way to answer: Examine what happens to "stuff" as it moves around near this region.

Specifically: Drop a particle from some finite starting radius r_0 (no angular momentum – purely radial trajectory).

Integrate the geodesic equation, calculate its position r as a function of "distant observer time t" and as a function its ***own*** time τ (also called "proper time".

The event horizon

Result:

$$t - t_0 = 2M \ln\left[\frac{(r/2M)+1}{(r/2M)-1}\right] - 4M\sqrt{\frac{r}{2M}}\left(1+\frac{r}{6M}\right) + C(r_0)$$

$$\tau - \tau_0 = \frac{4M}{3}\left(\frac{r_0}{2M}\right)^{3/2} - \frac{4M}{3}\left(\frac{r}{2M}\right)^{3/2}$$

Expand this around $r = 2M$: put $r = 2M + \delta r$,

$$\delta r = \tau_0 - \tau$$

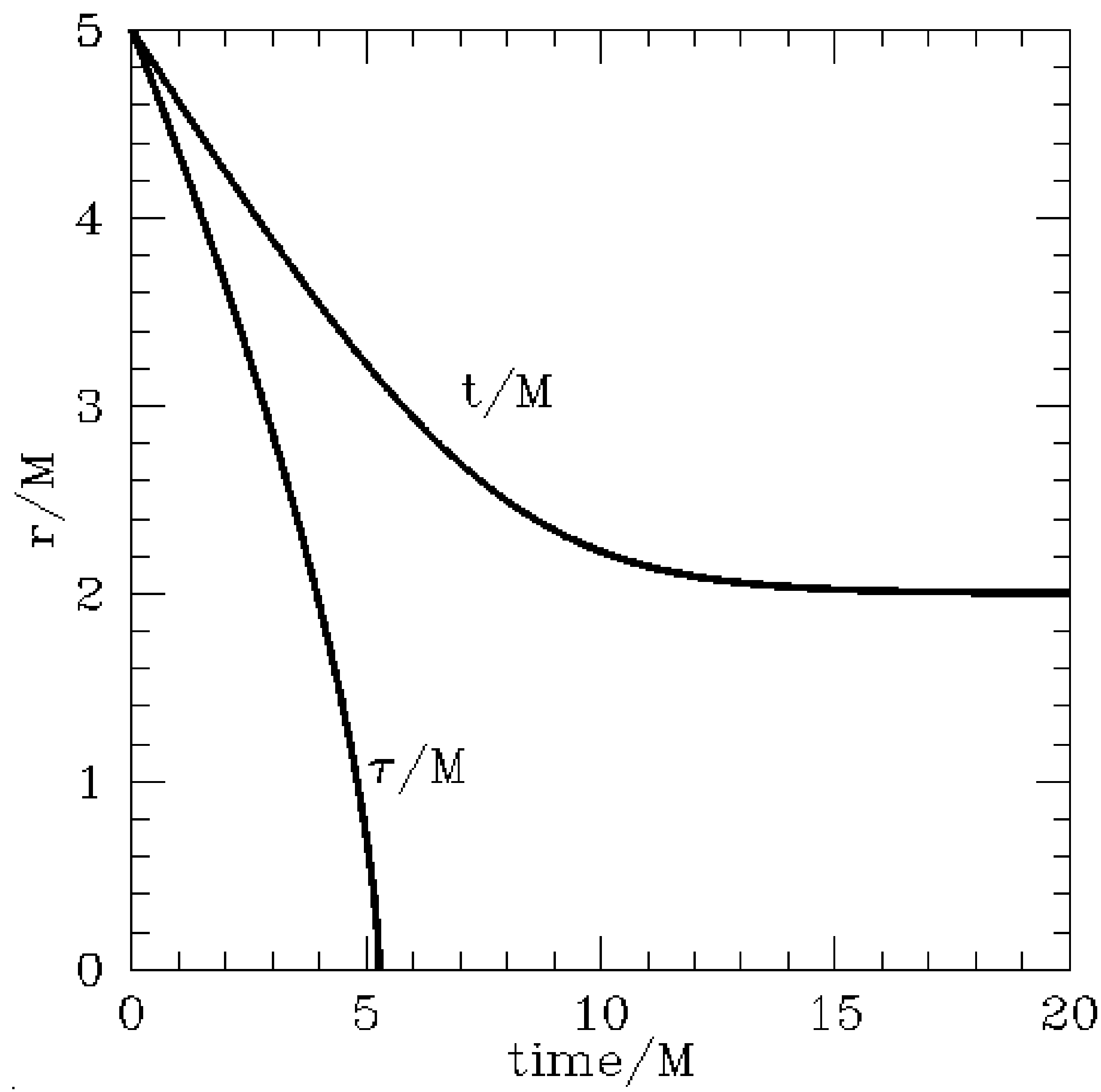
$$\delta r = 4Me^{-8/3}e^{(t_0-t)/2M}$$

Details: See Misner, Thorne, and Wheeler, *Gravitation*, Chapters 25 and 31.

The event horizon

A body that falls from finite radius passes through r = $2M$ in finite *proper* time, eventually reaching r = 0 …

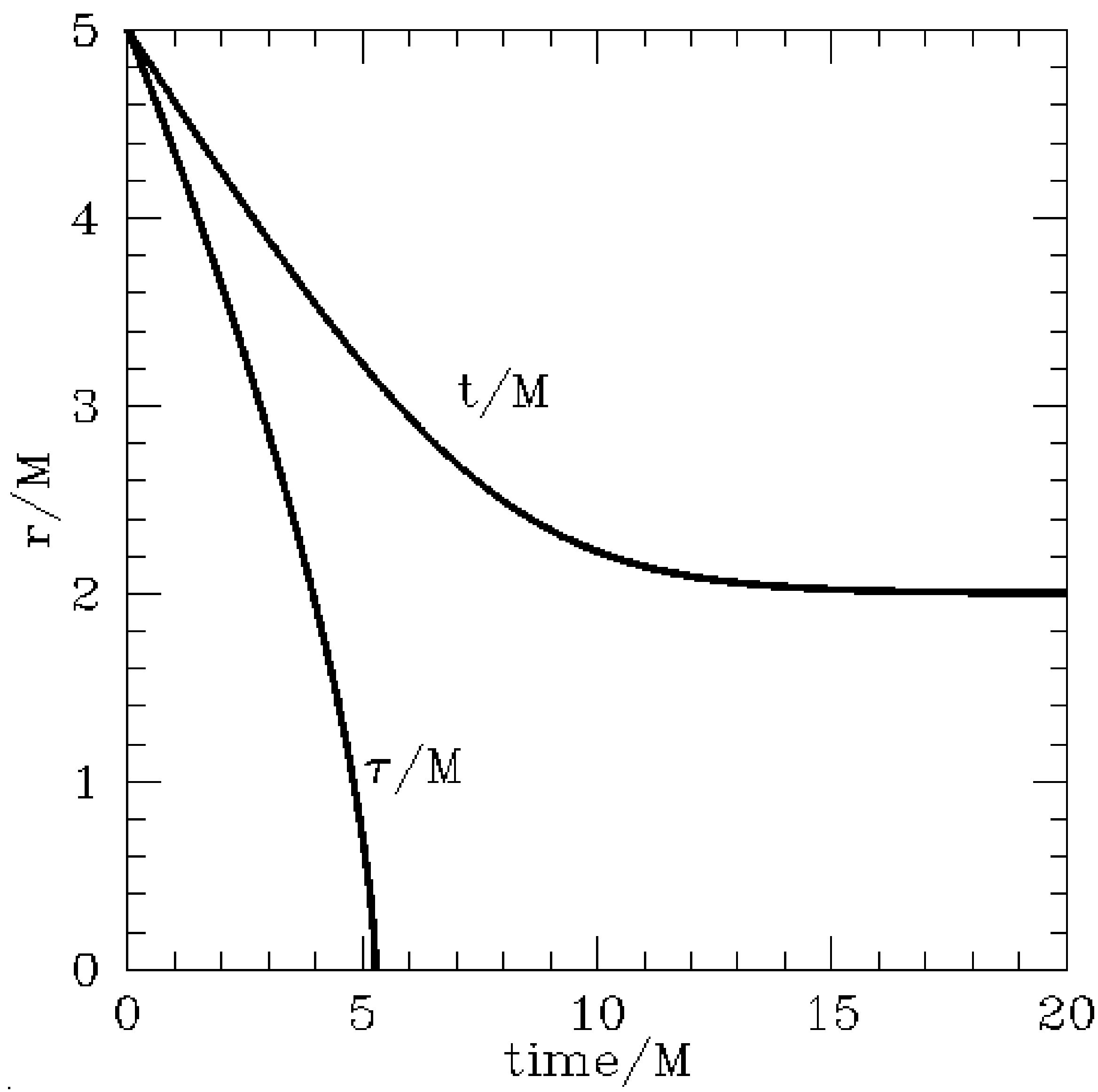
But, it takes ***infinite*** coordinate time to reach r = $2M$!!



The event horizon

This means that distant observers *never see the infalling body cross $r = 2M$.*

The infalling body itself passes through there with no problems - doesn't notice anything 'special'.



The event horizon

Gravitational force in GR goes over to *tidal* forces … if there is any *physical*, *local* effect that signifies something special about $r = 2M$, it must show up in tidal forces.

Riemann curvature tensor:

$$R_{\alpha\beta\gamma\delta} = \frac{A\,M}{r^3}$$

(Non-zero only for certain combinations of $\alpha,\beta,\gamma,\delta$.)

$$A = \pm 2,\ \pm 1,\ \text{or}\ 0$$

There is nothing special about the curvature at $r = 2M$!

However, $r = 0$ is clearly singular: Infinite curvature means infinite tidal forces.

The event horizon

Clarify by examining radial *null* geodesics, the trajectories followed by light rays:

$$ds^2 = 0 \longrightarrow dt = \pm \frac{dr}{1 - 2M/r}$$

$$\equiv \pm dr^*$$

$$\text{where} \quad r^* = r + 2M \ln\left(r/2M - 1\right)$$

Takes a **long time** (according to distant observers) for light to leak out from near 2*M*

Takes ***infinite*** time in the limit $r = 2M$!

The event horizon

Further insight: examine energy associated with the radial null geodesic.

Use the following rule: *The measured energy of a photon with 4-momentum* **p** *is given by*

$$E = -\mathbf{p} \cdot \mathbf{u}$$

where **u** *is the 4-velocity of an observer.*

Static observer at radius *r*:

$$u^a = \left[(1 - 2M/r)^{-1/2}, 0, 0, 0\right]$$

The event horizon

Result: For a photon emitted at some radius *r* and observed very far away,

$$\frac{E_{\rm obs}}{E_{\rm em}} = \sqrt{1 - \frac{2M}{r}}$$

Radiation emitted near $r = 2M$ is highly redshifted, with $r = 2M$ corresponding to a surface of ***infinite redshift***.

Coordinate pathologies are due to this infinite redshifting as we approach $r = 2M$.

The event horizon

To get rid of the pathologies, change to coordinates that aren't ill-behaved. Great example: *Kruskal-Szekeres* (KS) coordinates.

$$ds^2 = \frac{32M^2}{r} e^{-r/2M} \left(-dv^2 + du^2\right) + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

$$u = (1 - r/2M)^{1/2} \, e^{r/4M} \sinh(t/4M)$$

$$v = (1 - r/2M)^{1/2} \, e^{r/4M} \cosh(t/4M)$$

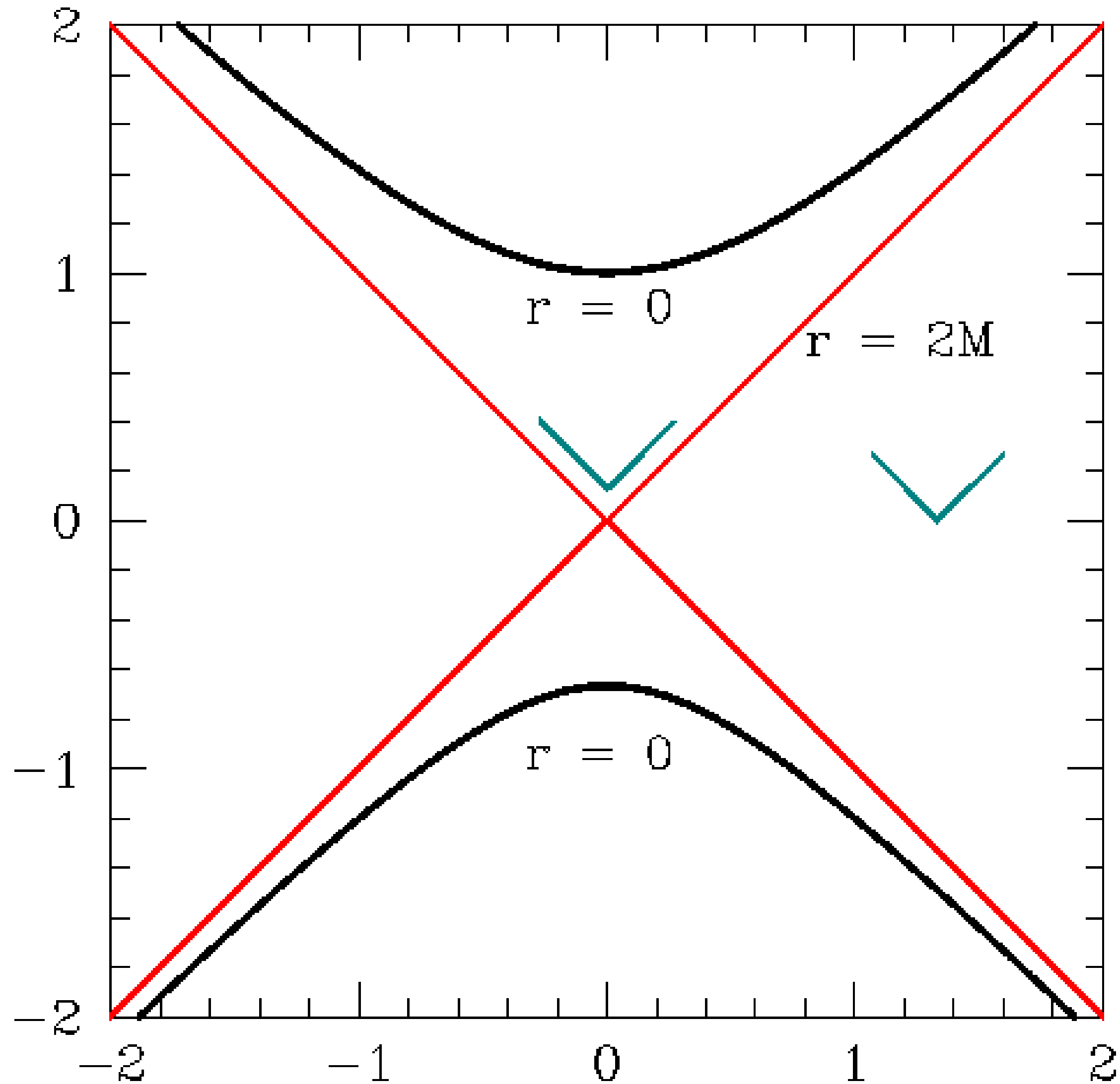
Radial motion of radiation is really simple in KS coords: just need *du* = +/- *dv*. In other words,

Light travels on 45 *degree lines in KS coordinates.*

The event horizon

Key thing this shows:
Nothing that is inside r = 2M can ever come out!
Even light is trapped inside this radius: Hence “horizon”!

All physical trajectories inside the horizon hit the singularity at *r = 0*.



Summary

The Schwarzschild metric is a perfectly viable candidate description for highly condensed objects in the universe.

- Exterior spacetime describes *any* spherically symmetric gravitating body.
- Event horizon at $r = 2M$ is odd, but reasonable: From exterior perspective, just the extreme limit of gravitational redshift.
- Reason that it's a horizon: Anything that passes inside $r = 2M$ cannot *ever* get out.

Schwarzschild from “normal” stuff?

Schwarzschild is mathematically viable ... but can we actually get it from reasonable initial conditions? Put it another way:

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Schwarzschild is mathematically viable … but can we actually get it from reasonable initial conditions? Put it another way:

Does smooth, normal "stuff" actually collapse into this odd solution??

Yes.

Reference: J. R. Oppenheimer and H. Snyder, "On Continued Gravitational Contraction", Phys. Rev. **56**, 455 (1939).

Oppenheimer-Snyder Collapse

Idealized but illustrative, totally *analytic* calculation of gravitational collapse.

Examine the collapse of a pressureless "dustball". Initial data is totally smooth. Ball's radius shrinks (increasing in density) due to self gravity.

Surface passes through $r = 2M$ without incident: A black hole is left behind.

Oppenheimer-Snyder Collapse

Some details of the calculation:

Use Friedman-Robertson-Walker metric to describe the interior (“collapsing universe cosmology”)

Use Schwarzschild metric to describe the exterior (Birkhoff’s theorem)

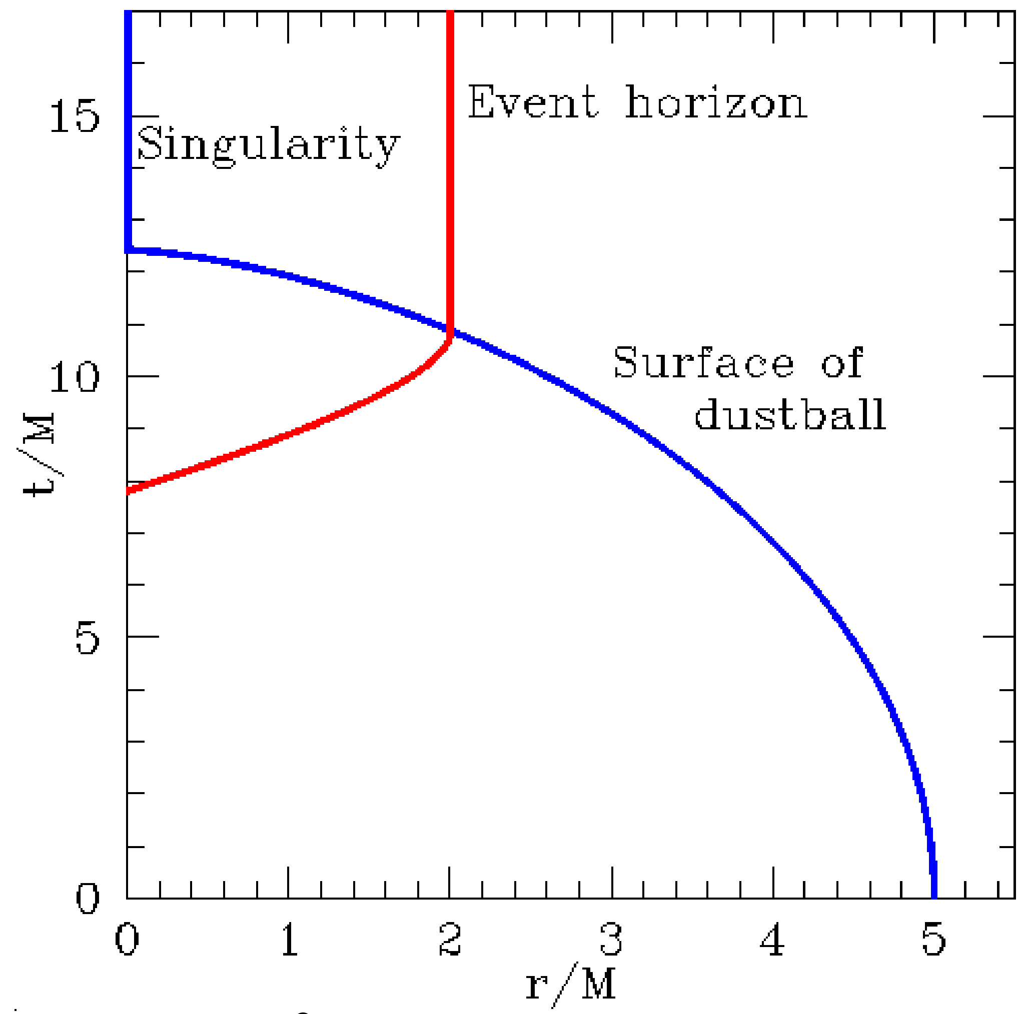
Match: Require that the metric components in the interior and the exterior forms agree at the surface.

O-S Collapse: More details

Spherical collapse with pressure can't be done analytically … but, can be done numerically without too much problem.

Always find that, if pressure is not sufficient, the object collapses through $r = 2M$ and leaves a black hole behind.

Oppenheimer-Snyder collapse



Black holes are a generic feature of spherical gravitational collapse!!

Life is not spherical

We should worry that this spherical description is ***too*** idealized!

Fortunately, a rotating black hole solution was discovered by Roy Kerr in 1963:

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4aMr\sin^2\theta}{\Sigma} dt\, d\phi + \frac{\Sigma}{\Delta} dr^2 \\
&+ \Sigma\, d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2 \\
a &= \frac{|\vec{S}|}{M} = \frac{G}{c}\frac{|\vec{S}|}{M}\,, \qquad \Delta = r^2 - 2Mr + a^2\,, \\
\Sigma &= r^2 + a^2\cos^2\theta
\end{aligned}
$$

Features of Kerr

- Reduces to Schwarzschild for $a = 0$. (Clear in these coordinates – "Boyer-Lindquist".)
- ***Not*** spherical! Constant r, constant t slice of a spherically symmetric spacetime can be covered by coordinates θ, ϕ with $g_{\phi\phi} = \sin^2\theta \, g_{\theta\theta}$. ***Cannot*** be done with Kerr (except for $a = 0$).
- Exhibits *frame dragging*: connection of t with ϕ (via $g_{\phi t}$ metric component) drags objects to whirl around black hole parallel to its spin.
- Has an event horizon at larger root of $\Delta = 0$:

$$r_{\mathrm{horiz}} = M + \sqrt{M^2 - a^2}$$

Unique black hole solution

The Kerr solution is the unique solution for astrophysical black holes: Any perturbation to the Kerr solution, large or small, is radiated away taking us back to the Kerr solution.

No Kerr analog of Oppenheimer-Snyder collapse exists; but, numerical calculations show that Kerr black holes are *the* result of final collapse.

Kerr black holes are the most plausible outcome of gravitational collapse in general relativity.