Unifying the Forces (and Particles!)

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Outline:

1. SM review: forces, particles, Higgs mechanism
2. Quick review of group theory:
   - representations, generators, roots & weights, etc.
3. The basic GUT idea
4. Choices of GUT groups & their relations
5. SM embedding into SU(5)
6. Gauge coupling unification & the GUT scale
7. Breaking the GUT group - hierarchy problem
8. Other issues and experimental signatures:
   - fermion masses
   - proton decay, B-L conservation
   - cosmological implications
   - extensions: SUSY, SO(10) & neutrinos, ...
9. Modern topics:
   - unification with gravity
   - embedding into string theory
   - GUTs in higher dimensions, TeV-scale GUTs
   - orbifold GUTs
SM review

Standard Model is the theory governing all fundamental particles and interactions for \( l \gtrsim 10^{-18} \text{ m} \iff E \lesssim 10^2 \text{ GeV} \).

It is a theory of FORCES & the PARTICLES on which they act. ("verbs") ("nouns")

We shall review only the grossest, "architectural" structure of the SM.

<table>
<thead>
<tr>
<th>FORCES (verbs)</th>
<th>&quot;electroweak&quot;</th>
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</thead>
<tbody>
<tr>
<td>strong ( \otimes )</td>
<td>weak ( \otimes )</td>
</tr>
<tr>
<td>( SU(3)_c ) &amp; ( SU(2)_W ) &amp; ( U(1)_Y )</td>
<td></td>
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<tr>
<td>(spin-1) (bosons):</td>
<td>8 gluons</td>
</tr>
<tr>
<td>( d_3 \approx \frac{1}{8.5} ) &amp; ( d_2 \approx \frac{1}{29.6} ) &amp; ( d_Y \approx \frac{1}{18.3} )</td>
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</tbody>
</table>

[measured at \( Z \)-scale \( \approx 90 \text{ GeV} \)]
Particle content (nouns)

- Particles are "Chiral Fermions":
  - **Fermions**: Dirac bispinor $\psi$
  - **Chiral**: definite handedness:
    $$\psi_L = \frac{1}{2} (1 - \gamma_5) \psi \quad \text{left}$$
    $$\psi_R = \frac{1}{2} (1 + \gamma_5) \psi \quad \text{right}$$
    Each is only two components.

Particle content of SM consists of three generations of chiral fermions:

**LEFT**: Electroweak doublet:

$$
\begin{align*}
(Y_d) & (s) & (t) \\
(Y_e) & (\nu_e) & (\nu_e)
\end{align*}
$$
Quarks: Each comes in three colors ($u, d, s, b$)
Leptons: No colors

**RIGHT**: All components are singlets:

$$
\begin{align*}
(u) & (c) & (t) \\
(d) & (s) & (b)
\end{align*}
$$
Quarks: Each in three colors
Leptons: No colors

\(\text{not yet discovered! Assume massless!}\)
Let's adopt a very succinct notation to describe the transformation properties of the particles with respect to the SM gauge symmetries:

[First generation only; others just repeat...]

\[
\text{SU}(3)_C \otimes \text{SU}(2)_W \otimes \text{U}(1)_Y
\]

<table>
<thead>
<tr>
<th>(Q_L = (u)_L)</th>
<th>((3, 2)_{1/3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L_L = (\nu)_L)</td>
<td>((1, 2)_{-1})</td>
</tr>
<tr>
<td>(u_R)</td>
<td>((3, 1)_{4/3})</td>
</tr>
<tr>
<td>(d_R)</td>
<td>((3, 1)_{-2/3})</td>
</tr>
<tr>
<td>(e_R)</td>
<td>((1, 1)_{-2})</td>
</tr>
<tr>
<td>(\nu_R)</td>
<td>((1, 1)_{0})</td>
</tr>
</tbody>
</table>

\[\text{Note: } u_R \text{ is still } 3, \text{ not } \bar{3} : \text{ different handedness of the same up quark!}\]
But it's somewhat awkward to deal with some fields listed as $L$, others $R$. ... obscures the relations between different representations.

Recall "charge conjugation" operation: (particle $\leftrightarrow$ antiparticle)

$$\Psi^c = i\gamma^2 \Psi^*$$

Then we observe:

$$\left(\Psi_R\right)^c = i\gamma^2 \left[ \frac{1}{2} \left( 1 + \gamma_5 \right) \Psi \right]^*$$

$$= \frac{i}{2} \gamma^2 \left( 1 + \gamma_5 \right) \Psi^*$$

$$= \frac{1}{2} \left( 1 - \gamma_5 \right) \left[ i\gamma^2 \Psi^* \right]$$

$$= \left(\Psi^c\right)_L$$

$\Rightarrow$ The conjugate of a right-handed component of a fermion is the left-handed component of the conjugate fermion!
Thus, if \( u_p = (3, 1)^{4/3} \)

then \((u_p)^c = (\overline{3}, 1)^{-4/3} = (u_c)^L\)

We can thus drop all "L" subscripts and write all fields in terms of left-handed components:

\[
\begin{align*}
Q : & \quad (3, 2)^{+1/3} \\
L : & \quad (1, 2)^{-1} \\
U^c : & \quad (\overline{3}, 1)^{-4/3} \\
D^c : & \quad (\overline{3}, 4)^{+2/3} \\
e^c : & \quad (1, 2)^{+2} \\
[ & \quad \gamma^c : (1, 1)^0. \quad \ldots \text{if it exists!}]
\end{align*}
\]
The SM also has an "adverb" — the Higgs sector.

\[
\begin{align*}
\text{SU}(3)_c \otimes \text{SU}(2)_W \otimes \text{U}(1)_Y \\
\text{Weak} \quad (A^+, A^0) \\
\text{Hypercharge} \quad (B) \\
\downarrow \langle \phi \rangle \text{ Higgs ver} \\
\text{U}(1)_{\text{EM}} \\
(\gamma = \text{photon})
\end{align*}
\]

How does this happen?

Higgs field = complex doublet of spin-0 Lorentz scalar:

\[
\phi = \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix}, \quad Y = 1 \\
(\text{chosen})
\]

Imagining

\[
V(\phi) = -\mu^2 \phi^+ \phi + \lambda (\phi^+ \phi)^2
\]

\[
\Rightarrow \text{Minimum at } v = \sqrt{\frac{\mu^2}{\lambda}} \approx 246 \text{ GeV}
\]

Parametrize Higgs in terms of deviations relative to new vacuum:

\[
\langle \phi \rangle = \exp \left[ -i \frac{\xi(x)}{\nu} \frac{\gamma^5}{\gamma} \right] \begin{pmatrix} 0 \\ \nu + \eta(x) \end{pmatrix}
\]
Thus, Higgs degrees of freedom are now

\[ \tilde{\xi}(x): (\tilde{\xi}^+, \tilde{\xi}^0) \text{ would-be Goldstone bosons} \]

\[ \eta(x): \text{ the physical Higgs} \]

\[ \tilde{\xi}^0: \text{ massless} \]

\[ \eta: \text{ massive} \]

So how does the Higgs mechanism actually work?
The Higgs Mechanism (schematically...)

\[ SU(3)_c \otimes SU(2)_W \otimes U(1)_Y \]

(gluons) \[ (A^\pm, A^3) \]

Unaffected \[ \Xi^3 \]

\[ W^\pm \] massive

\[ \begin{align*}
M_{W^\pm}^2 &= \frac{g_2^2 v^2}{4} \approx 83 \text{ GeV} \\
M_{Z^0}^2 &= M_{W^\pm}^2 / \cos^2 \theta \approx 91 \text{ GeV}
\end{align*} \]

where \[ \tan \theta_W = \frac{g_Y}{g_2} \]

\[ \sin^2 \theta_W = \frac{g_Y^2}{g_Y^2 + g_2^2} = \frac{d_Y}{d_Y + d_2} \approx 0.23 \]

Then \[ d_Y^{-1} = \alpha_{EM}^{-1} \cos^2 \theta_W \]

\[ d_2^{-1} = \alpha_{EM}^{-1} \sin^2 \theta_W \]

\[ \alpha_{EM}^{-1} \approx 127 \]

and we have \[ Q_{EM} = T_3 + \frac{Y}{2} \]
That's it!

Well, not really...

1. Yukawa couplings!

\[ L \sim y_{dd} \overline{Q}_L \phi d_R + y_{Qu} \overline{Q}_L (i \gamma_2 \phi^*) u_R \]

\[ + y_{Le} \overline{L}_L \phi e_R + y_{eR} \overline{L}_L (i \gamma_2 \phi^*) e_R, \]

Then our fermions gain Dirac masses

\[ m_i = |y_i| \langle \phi \rangle \]

\[ \Rightarrow m_i = \frac{|y_i| V^-}{\sqrt{2}} \]

only if neutrinos have Dirac masses
(2) **three generations!**

essentially the fermion structure repeats, but with one subtlety

— mixing between generations

**Quarks:**

\[
\begin{pmatrix}
(u) & (d) \\
(c) & (s) \\
(t) & (b)
\end{pmatrix}
\]

where

\[
\begin{pmatrix}
d' \\
s' \\
b'
\end{pmatrix} = \begin{pmatrix}
\text{Cabibbo} \\
\text{mass eigenstates}
\end{pmatrix} \begin{pmatrix}
d \\
s \\
b
\end{pmatrix}
\]

**Leptons:**

\[
\begin{pmatrix}
(\nu_e') & (\nu_\mu') & (\nu_\tau') \\
\nu_e & \nu_\mu & \nu_\tau
\end{pmatrix}
\]

where

\[
\begin{pmatrix}
\nu_e' \\
\nu_\mu' \\
\nu_\tau'
\end{pmatrix} = \begin{pmatrix}
3 \times 3 \\
\text{MNS matrix}
\end{pmatrix} \begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix}
\]

→ Yukawa couplings are matrices \( Y_{AB} \) in flavor space

... flavor physics!

CP violation, etc...

Very complicated, no deep understanding!
### Standard Model Summary

<table>
<thead>
<tr>
<th></th>
<th>$SU(3)_c \otimes SU(2)_W \otimes U(1)_Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Gauge Bosons</strong></td>
<td></td>
</tr>
<tr>
<td>Spin-1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\text{gluons}: (8, 1)$</td>
</tr>
<tr>
<td></td>
<td>$A^+ \otimes A^3$: $(4, 3)$</td>
</tr>
<tr>
<td></td>
<td>$B$: $(1, 1)$</td>
</tr>
<tr>
<td><strong>Matter</strong></td>
<td></td>
</tr>
<tr>
<td>Spin-$\frac{1}{2}$</td>
<td></td>
</tr>
<tr>
<td>All left-handed</td>
<td></td>
</tr>
<tr>
<td>$Q = (ud)$: $(3, 2)^{+ \frac{1}{3}}$</td>
<td></td>
</tr>
<tr>
<td>$L = (\nu e)$: $(4, 2)^{-1}$</td>
<td></td>
</tr>
<tr>
<td>$u^c$: $(\overline{3}, 1)^{-\frac{4}{3}}$</td>
<td></td>
</tr>
<tr>
<td>$d^c$: $(\overline{3}, 1)^{\frac{2}{3}}$</td>
<td></td>
</tr>
<tr>
<td>$e^c$: $(1, 1)^{+2}$</td>
<td></td>
</tr>
<tr>
<td>$[\gamma e^c$: $(1, 1)]$</td>
<td></td>
</tr>
<tr>
<td>$\phi$: $(1, 2)^{+1}$</td>
<td></td>
</tr>
<tr>
<td><strong>Ew Higgs</strong></td>
<td></td>
</tr>
<tr>
<td>Spin-0</td>
<td></td>
</tr>
</tbody>
</table>

**Note:** The table above summarizes the representations of the Standard Model particles under the gauge and weak gauge symmetries.
Question: What sets values of $Y$?

Note: $U(1)_Y$ is an abelian group $\Rightarrow$ any normalizations are allowed!

For fermion (matter) content:

Since $Q_{EM} = T_3 + \frac{Y}{2}$

and since we have measured $Q_{EM}$ experimentally, relative hypercharge assignments are fixed by experimental observations!

But is there a theoretical reason for these relative values of $Y$?

i.e., can we predict relations such as

$|\text{proton charge}| = |\text{electron charge}|$

on the basis of a physical principle?
Chiral (ABJ) triangle anomaly cancellation!
Anomalies spoil consistency of theory
at the quantum level — come from diagrams

... where our chiral fermions run in the loops.
Cancellation of each kind of anomaly diagram requires:

1. $\text{Tr}_3 Y = 0$ summed over all colored fermions
2. $\text{Tr}_2 Y = 0$ summed over all fermion doublets
3. $\text{Tr} Y^3 = 0$ summed over all fermions with $Y \neq 0$
4. $\text{Tr} Y = 0$ summed over all fermions

**Unique Solution Is The SM Solution** (or its rescaling)

$\Rightarrow$ relative $Y$-values are fixed $\Rightarrow$ Charge Quantization!

but overall normalization still unfixed.
For Higgs field $\phi$, situation is different.
Since $\phi$ not yet discovered, we don’t know its $Q_{EM}$.

However, assuming the same overall normalization as for the fermions, we still have $Q_{EM} = T_3 + \frac{Y}{2}$.

We then must choose

$$\phi : (1, 2)_{+1}$$

So that

$$\phi = (\phi^+, \phi_0) \leq Q_{EM} = +1$$

$$\phi_0 \leq Q_{EM} = 0$$

Why? Since bottom component $\phi_0$ gets very small, i.e., $\langle \phi_0 \rangle = \frac{v}{\sqrt{2}}$,
it must be electrically neutral ($Q_{EM} = 0$) so that EM is the remaining unbroken symmetry!
STANDARD MODEL — Things to remember:

1. lots of seemingly disconnected representations for gauge bosons & particle content

2. three independent gauge couplings \((g_3, g_2, g_4)\)
   - no predictions for \(g_i\)
   - not even predictions for their ratios such as \(\sin^2 \theta_W = g_1^2 / (g_1^2 + g_2^2)\)

3. particle representations are complex
   e.g., \(Q_L = (3, 2, 1) / 3\), but no \((\bar{3}, 2, -1) / 3\)!

4. overall normalization for \(Y\) *unfixed*
   [since U(1)_y abelian]
   even though relative \(Y\)-values are fixed
Higgs mechanism breaks

\[ \text{SU}(2)_W \otimes \text{U}(1)_Y \xrightarrow{<\phi> \neq 0} \text{U}(1)_{\text{EM}} \]

where \( \phi^0 \) is EM-neutral.

In general, the subgroup which survives is the subgroup with respect to which the field getting the non-zero VEV is neutral.

\[ \sin^2 \theta_W = \frac{g_Y^2}{g_Y^2 + g_2^2} \]

\[ M_{(W,Z)} \propto g_Y \]

\[ \alpha_{EM}^{-1} \approx 127 \]

\[ \alpha_Y^{-1} \approx 98.3 \]

\[ \alpha_2^{-1} \approx 29 \]

\( E \leq \nu \) one coupling

\( E > \nu \) two couplings

\( <\phi> \neq 0 \) phase transition
Particle representations treat baryons & leptons separately.

Also, they are not joined by Yukawa couplings [mixed Yukawa couplings would violate gauge invariance!]

\[ \Rightarrow \quad \text{In SM,} \]

- Baryon \# (B) conserved
- Lepton \# (L) conserved

Thus, e.g., the lightest baryon (= proton) is stable!

Note: B is actually broken by instanton effects \((\text{small})\)

\[ \text{L can be broken by } \gamma \text{ Majorana mass terms (if they exist).} \]
Some group theory

Let's think a bit more explicitly about the groups $\text{SU}(2)$, $\text{SU}(3)$ in the SM.

What do they mean?

Start with $\text{SU}(2) \cong \text{SO}(3)$ "rotation group" angular momentum!

Group has 3 generators:

\[
\begin{align*}
\cdot \quad & J_+ = J_1 + i J_2 \\
\cdot \quad & J_- = J_1 - i J_2 \\
\cdot \quad & J_z = J_3
\end{align*}
\]

with commutation relations

\[
\left[ J_i, J_j \right] = i \epsilon_{ijk} J_k \quad \leftrightarrow \quad \left\{ \begin{array}{l}
[ J_3, J_\pm ] = \pm J_\pm \\
[ J_+, J_- ] = 2 J_z
\end{array} \right.
\]

$\Rightarrow$ Generators don't commute $\Rightarrow$ group is "non-abelian"

$\Rightarrow$ only one generator can be diagonalized

$\Rightarrow$ choose it to be $J_z$
Then states can be chosen as eigenstates of $J_z$:

$J_z |m\rangle = m |m\rangle$

- Can indicate states graphically in terms of $m$:

  **SINGLET REP.**  $j=0$  

  **SPINOR REP.**  $j=1/2$  

  **VECTOR/ADJOINT**  $j=1$  

  :  $j=3/2$  

  \[ \text{[ALL REPRESENTATIONS ARE REAL]} \]

For each representation $j$, the dimension of rep $= 2j+1$.

$J^+ |m\rangle \sim |m+1\rangle \implies J^+$ raises $m$ by 1

$J^- |m\rangle \sim |m-1\rangle \implies J^-$ lowers $m$ by 1

$J_z |m\rangle = m |m\rangle \implies J_z$ preserves $m$, gives location

Actions are graphically represented as:

\[ \begin{align*}
  &\xleftarrow{J^-} 1  \quad \xrightarrow{J^+} 1  \\
  \text{Acts on } m
\end{align*} \]
In the actual SM case of $SU(2)_W$,

\[
\begin{align*}
M & \rightarrow T_3 \\
J^\pm & \rightarrow A^\pm \\
J^Z & \rightarrow A^Z
\end{align*}
\]

Thus, the gauge bosons of $SU(2)_W$ have the actions:

\[ \begin{array}{c}
A^- \\
A^0 \\
A^+ \\
T_3
\end{array} \]

Moreover, as we saw, all fermion representations in the SM are either singlets or doublets:

\[ \begin{array}{c}
\text{e.g.,} \\
\text{e, u, d} \\
\text{d} \\
\text{u} \\
\text{e} \end{array} \]

So gauge bosons act according to these $T_3$ "charges":

\[ \begin{array}{c}
(T_3 = \pm \frac{1}{2}) \\
(A^+) \\
(T_3 = \pm \frac{1}{2}) \\
A^- \end{array} \]
**SU(2) SUMMARY**

- **one diagonal generator:** $J^2 |m\rangle = m |m\rangle$
- **three generators total ($J_+$, $J_-$):**

```
          -1  Jz  1
          -  0  1
          J-  J+  m
```

- **Representations:**
  - **Singlet**
    - $|0\rangle \rightarrow m$
  - **Doublet**
    - $|\pm \sqrt{1}/2\rangle \rightarrow m$
  - **Triplet**
    - $|1, 0, \pm 1\rangle \rightarrow m$
  - **Quartet**
    - $|\pm 3/2, \pm 1/2, \pm 1/2, \pm 3/2\rangle \rightarrow m$
  - **Etc.**

- All reps are real (i.e., same if inverted through origin, $m \rightarrow -m$)
- Generator diagram is same points as triplet representation

  $\Rightarrow$ triplet is "adjoint" representation.
Now consider \( \text{SU}(3) \)

Examples:
- \( \text{SU}(3) \) color \((r, g, b)\) (LOCAL)
- \( \text{SU}(3) \) flavor \((u, d, s)\) (GLOBAL)

Very different physically, but exactly the same algebraically \(\Rightarrow\) We shall use both examples!

\( \text{SU}(3) \): has eight generators \( T^1, T^2, \ldots, T^8 \)

E.g., in color case \(\Rightarrow\) 8 gluons
\(\Rightarrow\) 8 Gell-Mann matrices
\( g_i = 1, \ldots, 8 \)

**ONLY TWO CAN BE SIMULTANEOUSLY DIAGONALIZED** \(\Rightarrow\) Usually called \( T^3, T^8 \)

\(\Rightarrow\) All states can be chosen as eigenstates of \( T^3, T^8 \):

\[ T^3 | m_3, m_8 \rangle = m_3 | m_3, m_8 \rangle \]
\[ T^8 | m_3, m_8 \rangle = m_8 | m_3, m_8 \rangle \]

States can now be represented graphically in an \((m_3, m_8)\) PLANE:
Just as for SU(2), only certain representations are allowed for self-consistency:

- "singlet" \(1\)
- "triplet" \(3\) (fundamental)
- "antitriplet" \(\bar{3}\)
- Conjugate: flip each point through origin!
- "hextet" \(6\)
- "octet" \(8\) (real)
  \[8 = \bar{8}\]
- "decuplet" \(10\)

...etc.
e.g., in flavor case, we have

\[ \begin{array}{c}
\text{quarks} \\
\text{anti-quarks}
\end{array} \]

Thus, "adding" these together in various combinations (i.e., taking the tensor product \( 3 \times \bar{3} \)) we obtain

\[ d \bar{s} = K^0 \]
\[ u \bar{s} = K^+ \]
\[ k^- = \bar{u}s \]
\[ \eta^0 = \frac{1}{\sqrt{6}} (u \bar{u} + d \bar{d} - 2 s \bar{s}) \]

plus a singlet:

\[ u \bar{u} + d \bar{d} + s \bar{s} \]
It turns out that all larger reps can be constructed in this way by tensoring together combinations of the fundamental representations:

- e.g. $3 \otimes \bar{3} = 8 \oplus 1$ (MESONS)

$3 \otimes 3 = 6 \oplus 3$

$6 \otimes 3 = 10 \oplus 8$

$3 \otimes 3 \otimes 3 = 10 + 8 + 8 \oplus 1$ (BARYONS)

Big representations are just products of small representations!
These relations can also be represented by matrices.

Imagine the fundamental rep as a vector

\[ B_i = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad \overline{B}_i = \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} \]

Then, e.g.,

- \[ B_i \otimes B_i = \begin{pmatrix} M_{ji} \end{pmatrix} \quad \text{symmetric} \quad \text{anti-symmetric} \]

\[ j = (uds) \]

- \[ \overline{B}_j \otimes B_i = \begin{pmatrix} M_{ji} \end{pmatrix} \quad \text{traceless} \quad \text{trace} \]

\[ j = (\bar{u}\bar{d}\bar{s}) \]

where

\[ 8 = \begin{bmatrix} \bar{u}u & \bar{u}d & \bar{u}s \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{bmatrix} - \frac{1}{3} \left( \bar{u}\bar{u} + \bar{d}\bar{d} + \bar{s}\bar{s} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} \frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta^0 & \pi^+ & \pi^- \\ \pi^- & \frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta^0 & K^+ \\ K^- & \bar{K}^0 & \frac{1}{\sqrt{2}} \eta + \frac{1}{\sqrt{6}} \eta' \end{pmatrix} \]
For SU(3), there are 8 generators:

- $T^3, T^8$ ← analogues of $J_2$ for SU(2)
- six others ← analogues of $J_+$

These do "raising" and "lowering" in the two-dimensional $(m_3, m_8)$ plane:

For color $g^r \nabla_b$
these generators are all gluons

$g \bar{b}$
$g \bar{r}$
$r \bar{b}$
$r \bar{g}$
$b \bar{r}$
$b \bar{g}$

$T^3 = \frac{g \bar{g} - g \bar{r}}{v^2}$
$T^8 = \frac{r \bar{r} + g \bar{g} - 2b \bar{b}}{\sqrt{6}}$

Gluons act on quark lines as we expect, e.g.,

$g \bar{r} → T g$
$\bar{r} → r$
Aside: Why is it called $SU(3)$?

We have already seen that the generators are all $(3 \times 3)$-dimensional traceless Hermitian matrices

$$\begin{bmatrix}
\bar{r}r - T/3 & \bar{r}g & \bar{r}b \\
\bar{g}r & \bar{g}g - T/3 & \bar{g}b \\
\bar{b}r & \bar{b}g & \bar{b}b - T/3
\end{bmatrix}$$

$$\bar{a} = (\bar{r}g \bar{b})$$

where: $T = \bar{r}r + \bar{g}g + \bar{b}b$

These Hermitian generators are like operators $\hat{A}$
Recall that such operators can be exponentiated to form group elements

for $SU(2)$: e.g., $D(\Phi, \hat{\pi}) = \exp \left[ i \frac{\hat{\pi}}{\hbar} \Delta \Phi \right]$ → unitary rotation operator!

In general, group elements are

$$U(\epsilon a) = \exp \left[ i \frac{\epsilon}{a} \epsilon a T_a \right]$$

$\epsilon_a$: parameter

$T_a$: generators

Since $T^a$ are Hermitian & traceless $(3 \times 3)$
$\Rightarrow$ $U$ are unitary with det = 1 $(3 \times 3)$
$\Rightarrow$ $SU(3)$!
SU(3) summary:

- two diagonal generators
  \[ T_3 |m_3, m_8\rangle = m_3 |m_3, m_8\rangle \]
  \[ T_8 |m_3, m_8\rangle = m_8 |m_3, m_8\rangle \]

- together, all generators fill out the pattern

- with \( T^3 \) in the center,
  (do not raise or lower)

- SU(3) representations are

\[ \begin{align*}
1 & \quad 3 \\
3 & \quad 6 & \quad 8 & \quad 10
\end{align*} \]

Note: 1, 8 are real
3, 6, 10 are complex ...
(8 is adjoint)
In general, these types of patterns continue to \textbf{LAGER GROUPS} as well.

In each case,

- \# of commuting generators (like $T^3$, $T^8$) \equiv \text{dimensionality of our plots} \equiv \text{RANK of group}

- total \# of generators \equiv \text{ORDER of group}

- generators fill out pictures

\begin{align*}
  \begin{array}{c}
  \text{su}(2) \\
  \text{su}(3)
  \end{array}
\end{align*}

\begin{align*}
  \begin{array}{c}
  \text{these vectors are called "roots".} \\
  \text{These dots are called "weights."}
  \end{array}
\end{align*}

- representations are states which fill out dot patterns

\begin{align*}
  \begin{array}{c}
  \text{etc.}
  \end{array}
\end{align*}

We can then classify all possible groups!
### Complete Classification of Lie Groups:

<table>
<thead>
<tr>
<th>Name</th>
<th>Rank</th>
<th>Serial Number (Order)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SU(n)</td>
<td>n-1</td>
<td>(n^2-1)</td>
</tr>
<tr>
<td>SO(2n+1)</td>
<td>n</td>
<td>(n(2n+1))</td>
</tr>
<tr>
<td>Sp(2n)</td>
<td>n</td>
<td>(n(2n+1))</td>
</tr>
<tr>
<td>So(2n)</td>
<td>n</td>
<td>(n(2n-1))</td>
</tr>
<tr>
<td>E_6</td>
<td>6</td>
<td>78</td>
</tr>
<tr>
<td>E_7</td>
<td>7</td>
<td>133</td>
</tr>
<tr>
<td>E_8</td>
<td>8</td>
<td>248</td>
</tr>
<tr>
<td>F_4</td>
<td>4</td>
<td>52</td>
</tr>
<tr>
<td>G_2</td>
<td>2</td>
<td>14</td>
</tr>
</tbody>
</table>

E.g., \(SU(4)\): rank 3; 15 generators \((T^3, T^8, T^1^5)\) diagonal.

Plot representations in a 3-dimensional space:
SUBGROUPS

\[ \text{e.g., } SU(3) \rightarrow SU(2) \otimes U(1) \]

"flavor" "isospin" \[ Y \]

\[ \gamma = \text{weak isospin} \]
\[ = \text{strangeness + const.} \]

Just "decompose" representations!

\[ Y \]
\[ \text{axis} \]

\[ SU(3) \times U(1)_Y \]

\[ SU(2) \text{ axis} \]

\[ \begin{array}{c}
3 \\
\rightarrow \\
\left( \frac{2}{2 \sqrt{3}} \right) \oplus \left( \frac{1}{-3 \sqrt{3}} \right)
\end{array} \]

We can rescale \( Y \) to eliminate fractions if we wish:

\[ 3 \\
\rightarrow \\
\left( 2 \right)_{\frac{1}{2 \sqrt{3}}} \oplus \left( 1 \right)_{\frac{-1}{3 \sqrt{3}}} \]

TWO REPRESENTATIONS of subgroup, \( SU(2) \otimes U(1) \),

when properly chosen,

combine into a single representation of a bigger group, \( SU(3) \)!
eg, adjoint rep of $SU(3)$:

$$8 \rightarrow (2)_{+1} \oplus (2)_{-1} \oplus (3)_{0} \oplus (1)_{0}$$

$\text{real}$

$\text{conjugates}$

$\text{real}$

$\text{real}$

eg, decuplet

$$10 \rightarrow (4)_{+1} \oplus (3)_{0} \oplus (2)_{-1} \oplus (1)_{-2}$$

If $H$ is a subgroup of $G$, then every rep of $G$ can decompose into sums of reps of $H$!
Or, going backwards, sometimes
a set of reps of one group,
when properly chosen with all quantum
turns properly balanced, can
combine to fill out a
SINGLE REP of a LARGER GROUP!

Hmmm...

That was our original goal!
-to "unify" all of the forces and particles!

We now see how to do this:

⇒ We need a bigger group!

(and hope for a few
miracles along the way...)
What groups $G$ can we choose?

Requirements:

1. SM has rank $= 4$ (four commuting generators) \[ T_3, T_8 \quad A_3, B \quad \text{gluons} \quad 2^0, 8 \] \[ \implies \] group $G$ must be rank $\geq 4$ and contain SM as subgroup

2. SM has complex representations \[ \text{eg (3, 2), } T_3, \text{ no } (\bar{3}, 2), -T_3 \] \[ \implies \] group $G$ must also have complex reps \[ \text{eg } SU(6) \text{ } \text{doesnt } \quad \text{Su(3) does} \]

3. SM is free of chiral anomalies \[ \implies \] group $G$ must have reps for which anomalies can cancel

4. If we want to relate the couplings \( g_1, g_2, g_3 \) to each other \[ \implies \] $G$ should be a simple group \( (\text{not a product of different, unrelated factors}) \)
Look back at previous list of groups.
What are our options?

<table>
<thead>
<tr>
<th>Rank</th>
<th>Group</th>
<th>Group</th>
<th>Group</th>
<th>Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>U(1)</td>
<td>SU(2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>SU(3)</td>
<td>SO(5)</td>
<td></td>
<td>G_2</td>
</tr>
<tr>
<td>3</td>
<td>SU(4)</td>
<td>SO(7)</td>
<td>Sp(6)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>SU(5)</td>
<td>SO(9)</td>
<td>Sp(8)</td>
<td>F_4</td>
</tr>
<tr>
<td>5</td>
<td>SU(6)</td>
<td>SO(11)</td>
<td>Sp(10)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>SU(7)</td>
<td>SO(13)</td>
<td>Sp(12)</td>
<td>E_6</td>
</tr>
<tr>
<td>7</td>
<td>SU(8)</td>
<td>SO(15)</td>
<td>Sp(14)</td>
<td>E_7</td>
</tr>
<tr>
<td>8</td>
<td>SU(9)</td>
<td>SO(17)</td>
<td>Sp(16)</td>
<td>E_8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

〇 CIRCLE INDICATES GROUP HAS COMPLEX REPS.
Therefore, our options are:

- if $G_1 =$ simple, then $G = \begin{cases} SU(5) \text{ rank 4} \\ SU(6), SO(10) \text{ rank 5} \\ SU(7), E_6 \text{ rank 6} \end{cases}$

- if $G_1 =$ product group, then
  - require complex factor to contain $SU(3)$ as subgroup
  - remaining factors to contain $SU(2) \times U(1)$

$\Rightarrow G = \begin{cases} SU(3) \times SU(2) \times SU(2) \text{ rank 4} \\ SU(3) \times SU(3) \\ SU(3) \times SO(5) \end{cases}$

$\begin{cases} SU(4) \times SU(2) \times U(1) \text{ rank 5} \\ SU(4) \times SU(2) \times SU(2) \\ SU(4) \times SU(3) \\ SU(4) \times SO(5) \\ SU(3) \times SO(7) \\ SU(3) \times Sp(6) \end{cases}$

\[\vdots\text{ etc.}\]
Most of these choices do not succeed in producing interesting unifications. However, interesting cases are:

rank - 4: \( SU(5) \)

Pati-Salam

rank - 5: \( \left\{ \begin{array}{l} SO(10) \\ SU(4) \times SU(2) \times SU(2) \end{array} \right. \)

rank - 6: \( E_6 \)

These groups have the relative subgroup structure:

\[ E_6 \rightarrow SO(10) \otimes U(1) \]

\[ \begin{array}{c}
SU(5) \otimes U(1) \\
SU(4) \otimes SU(2) \otimes SU(2) \otimes U(1)
\end{array} \]

Let's begin by looking at \( SU(5) \).
\[
\begin{align*}
\text{SU}(5) & \rightarrow \text{SU}(3) \otimes \text{SU}(2) \otimes U(1)
\end{align*}
\]

Representations decompose as

\[
\begin{align*}
1 & \rightarrow (1, 1) , \\
5 & \rightarrow (3, 1)_2 \oplus (1, 2)_3 \\
10 & \rightarrow (3, 2)_1 \oplus (\overline{3}, 1)_4 \oplus (1, 1)_6 \\
15 & \rightarrow (6, 1)_4 \oplus (3, 2)_1 \oplus (1, 3)_6 \\
24 & \rightarrow (8, 1)_6 \oplus (3, 2)_5 \oplus (\overline{3}, 2)_5 \\
& \quad \oplus (1, 3) , \oplus (1, 1)
\end{align*}
\]

30, 40,
45, 50, 70 etc...

where all \( U(1) \) charges are normalized to avoid fractions
(as is conventionally done in standard references...)

Does this have the potential for a Successful Unification?
Forms of SM:

Recall each SM generation contains 15 states

\[
\begin{pmatrix}
\begin{array}{c}
(u) \\
(d) \\
(u^c) \\
(d^c)
\end{array}
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
(12) \\
(1) \\
(12)^c
\end{array}
\end{pmatrix}
\times 3 \text{ colors}
\]

However, the 115 representation of SU(5) does not accommodate them

(e.g., 115 gives a color sextet!)

\[\text{Nor does the 45 representation accommodate the three generations.}\]

But look at 110 - if we rescale above U(1) quantum field by \(\frac{1}{3}\), it becomes

\[
110 \rightarrow \begin{pmatrix}
\begin{array}{c}
(3, 2)^{4/3} \\
(\bar{3}, 1)^{-4} \\
(1, 1)^{2}
\end{array}
\end{pmatrix}
\]

All that's left is

\[
d^c = (\bar{3}, 1)^{2/3} \quad \text{and} \quad L = (1, 2)^{-1} \}
\]

\(5\) states

⇒ These don't fit into a \(5\)

but into a \(\bar{5}\)!
Thus, an entire SM generation fits into \[ \overline{5} + 10 \] of \( SU(5) \) with nothing left over (no exotics)!

In matrix notation, this is

\[
\overline{\mathbf{5}} = \begin{pmatrix} d^c & d^c & d^c \\ u^c & Q \\ \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \end{pmatrix}
\]

\[ \text{row vector} \]

\[ \text{color part} \quad \text{weak part} \]

\[ 10 = \text{antisymmetric component of } 5 \times 5 \text{ matrix} \]

\[
\begin{pmatrix}
0 & u^c & u^c & u^c & dr \\
0 & 0 & u_g & u_g & dg \\
0 & 0 & 0 & u_b & db \\
0 & 0 & 0 & 0 & e^c \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where (recall) \[ \begin{cases} 
\mathbf{u}^c: (\overline{3}, 1) & SU(3) \text{ triplet, } SU(2) \text{ singlet} \\
\mathbf{Q}: (3, 2) & \text{ triplet, doublet} \\
\mathbf{e}^c: (1, 1) & \text{ singlet, singlet} 
\end{cases} \]
24 → \((8,1) \oplus (1,3) \oplus (1,1) \oplus (3,2) \oplus (\bar{3},2)\)

- gluons: \(A^t, A^3, B\)
- SM gauge bosons all successfully embedded!

Proof that:
\[ \text{SU}(3) \otimes \text{SU}(2) \otimes \text{U}(1) \subseteq \text{SU}(5) \]

But what are these??!

Appear to be gauge bosons carrying:
- color
- weak charge
- hypercharge ... simultaneously!

Also,
\[ Q_{EM} = T_3 + \frac{Y}{2} = \{ \pm \frac{1}{3}, \pm \frac{4}{3} \} \]

They are also electromagnetically fractionally charged as well!
In matrix language:

Since $\bar{\psi} \psi = 24 + 11$

This is a traceless matrix:

\[ 24 = \left( \begin{array}{cc|c|c}
(\text{quarks}) & X_\tau & Y_\tau & 1 \\
(\text{gluons}) & X_g & Y_g & X_b & Y_b \\
X_\tau & X_g & X_B & \left( A^+, A^3, B \right) \\
Y_\tau & Y_g & Y_B & &
\end{array} \right) \]

\[ Q_{\text{EM}}^{(x)} = -\frac{1}{3} \]
\[ Q_{\text{EM}}^{(y)} = -\frac{4}{3} \]
\[ Q_{\text{EM}}^{(y)} = \frac{1}{3} \]
\[ Q_{\text{EM}}^{(y)} = \frac{4}{3} \]

⇒ "X, Y" gauge bosons are "off-diagonal" in color/electroweak space, carry both types of charges simultaneously!

⇒ They can connect quarks to leptons!
e.g.,

\[ X_r \quad (Q = -\frac{1}{3}, T_3 = \frac{1}{2}) \]

or

\[ Y_r \quad (Q = -\frac{4}{3}, T_3 = -\frac{1}{2}) \]

In each case:
- \( Q_{EM} \) is conserved
- \( T_3 \) is conserved
- Color lines flow correctly

\( X, Y \) appear as "leptoquarks" - injecting color and \( T_3 = \pm \frac{1}{2} \)!

\[ \Rightarrow \] Baryon \# and lepton \# no longer conserved!

Each of these diagrams is \( AB = \frac{1}{3} \) process! ("LEPTOQUARK CHANNEL")
There is also another channel in which \((X,Y)\) bosons can act:

Since \((X,Y)\) carry \(\overline{3}\) of color, they can also turn quarks directly into antiquarks because \(3 \otimes \overline{3} = 6 \oplus \overline{3}\)

\[
\begin{align*}
Q &= -\frac{1}{3} \\
T_3 &= \frac{1}{2}
\end{align*}
\]

\(X_r\) \(\rightarrow\) \(\overline{U}\) \(\rightarrow\) \(\bar{d}\) \(\rightarrow\) \(\overline{U_g}\) \(\rightarrow\) \(U_g\)

\[
\begin{align*}
Q &= -\frac{2}{3} \\
T &= 0
\end{align*}
\]

\[
\begin{align*}
Q &= -\frac{1}{3} \\
T &= -\frac{1}{2}
\end{align*}
\]

\[
\begin{align*}
Q &= -\frac{4}{3} \\
T_3 &= -\frac{1}{2}
\end{align*}
\]

\(Y_r\) \(\rightarrow\) \(\overline{U}\) \(\rightarrow\) \(\bar{d}\) \(\rightarrow\) \(\overline{U_g}\) \(\rightarrow\) \(U_g\)

\[
\begin{align*}
T &= 0 \\
Q &= -\frac{2}{3}
\end{align*}
\]

\[
\begin{align*}
T &= \frac{1}{2} \\
Q &= \frac{2}{3}
\end{align*}
\]

In such channels, \((X,Y)\) act as "di-quarks", not leptogluarks!

Here \(\Delta B = -\frac{2}{3}\) since \(q \rightarrow \overline{q}\).
Note that in each channel,

(X,Y) gauge bosons change

\underline{incoming \ particles}\n
into

\underline{outgoing \ antiparticles}!

Very strange...

\underline{fermion \ # \ is \ not \ conserved}!
Finally, the Higgs representation $\phi : (1, 2)_1$ is embedded easily into the 5 rep of $SU(5)$:

$$\begin{align*}
5 &\rightarrow (3, 1)_{-2/3} \oplus (1, 2)_1 \\
\text{usual Higgs doublet } q &
\end{align*}$$

A new "colored" Higgs triplet $\phi_3$,

$$Q_{EM} = T_3 + \frac{Y}{2} = -\frac{1}{3}$$

[same quantum #'s as a RH down quark $d_R$]

This also mediates new interactions because of the Higgs Yukawa couplings.

E.g., $\mathcal{L}_{SM} = Y_d \overline{d_L} \phi \phi \phi \rightarrow \mathcal{L}_{SU(5)} = Y \overline{T_0 \cdot 10 \cdot 5 \cdot 5}$ contains a singlet

E.g.,

$$\begin{align*}
(T_3 = 0) \\
Q_{EM} = -\frac{1}{3}
\end{align*}$$

$\Delta B = \frac{1}{3}$