Intro to General Relativity

SSI 2005 - Sean Carroll

Metric : $g_{\mu\nu}(t, \vec{x})$

Geodesics : $x^\mu(\lambda)$

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \qquad \left(\sum_{\rho,\sigma}\right)$$

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\lambda}\left(\partial_\rho g_{\sigma\lambda} + \partial_\sigma g_{\lambda\rho} - \partial_\lambda g_{\rho\sigma}\right)$$

Riemann curvature:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\sigma\mu}$$

Ricci Tensor: $R_{\sigma\nu} = R^\lambda{}_{\sigma\lambda\nu}$

Curvature scalar: $g^{\mu\nu}R_{\mu\nu} = R$

Einstein's Eq: $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$

$T_{\mu\nu}$ = energy - momentum tensor

Outline

- Overview
- GR is a field theory
- Equivalence principle
- The metric
- Geodesics
- Spacetime
- Non-flat metrics
- Vectors
- Tensors
- Non-tensors
- Covariant derivatives
- Gauge theory
- Curvature
- Einstein's equation
- Matter
- Newtonian limit
- Cosmology
- Homework

GR is a field theory

2 ways to get GR:

- "Gravity" = curvature of space time

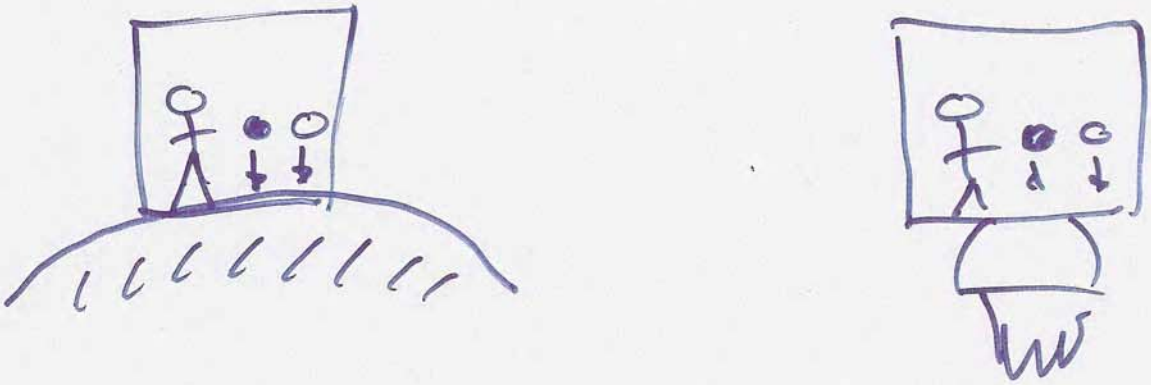
  metric tensor: $g_{\mu\nu}(t,\vec{x}) = g_{\nu\mu}$

- massless spin-2 particles

  $h_{\mu\nu} = h_{\nu\mu}$

  $\rightarrow g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h^2)$

Principle of Equivalence

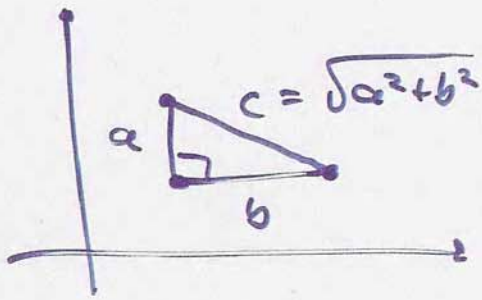


→ Gravity is universal

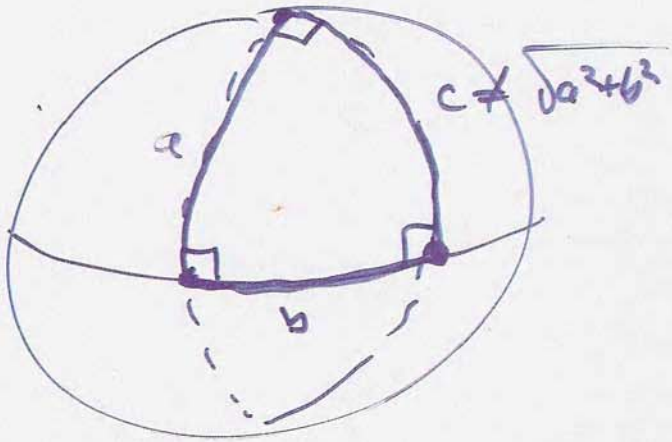
→ ∴ it is not a "force"

Metric

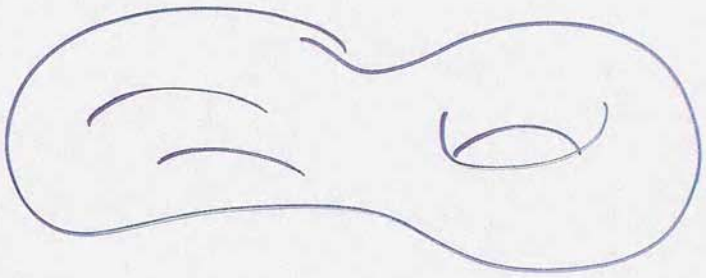
Flat:



Curved:



Curvature need not be uniform:



Go infinitesimal!

Curvature:

$$ds^2 = f(x,y)\,dx^2 + g(x,y)\,dx\,dy + h(x,y)\,dy^2$$

Better notation:

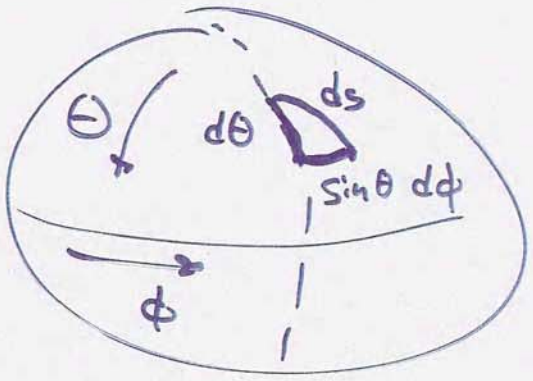
$$ds^2 = \sum_{i,j} g_{ij}(x)\,dx^i\,dx^j$$

$$x^i = \{x, y\}$$

$$ds^2 = g_{ij}\, dx^i dx^j$$

→ line element

Sphere:

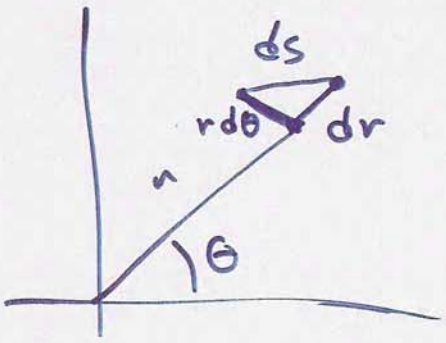


$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

$$= g_{ij}\, dx^i dx^j$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

Plane in polar coordinates:

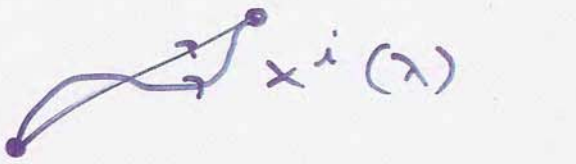


$$ds^2 = dr^2 + r^2 d\theta^2$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Geodesics : minimum-length paths

Integral:



$$I = \int ds = \int \sqrt{ds^2} = \int \sqrt{g_{ij}\, dx^i\, dx^j}$$

"Velocity": $v^i = \frac{dx^i}{d\lambda}$

$$\left(\frac{ds}{d\lambda}\right)^2 = g_{ij}\left(\frac{dx^i}{d\lambda}\right)\left(\frac{dx^j}{d\lambda}\right) = g_{ij}\, v^i v^j = \vec{v}\cdot\vec{v}$$

$$I = \int \sqrt{g_{ij}\, \frac{dx^i}{d\lambda}\, \frac{dx^j}{d\lambda}}\; d\lambda$$

Vary the path: $x^i(\lambda) \to x^i(\lambda) + \delta x^i($

$$I \to I + \delta I$$

$$\frac{dx^i}{d\lambda} \to \frac{dx^i}{d\lambda} + \frac{d(\delta x^i)}{d\lambda}$$

$$g_{ij}(x^k) \to g_{ij}(x^k + \delta x^k)$$

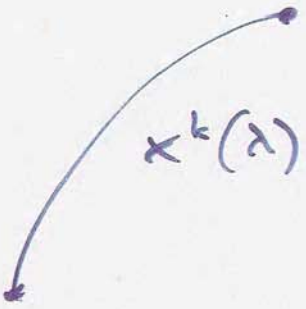
$$= g_{ij}(x^k) + \frac{\partial g_{ij}}{\partial x^k}\, \delta x^k$$

$$= g_{ij}(x^k) + \partial_k g_{ij}\, \delta x^k$$

$$I = \int ds = \int \sqrt{g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}\, d\lambda$$

$$x^i \to x^i + \delta x^i$$

$$I \to I + \delta I$$



$$\delta I = 0$$

$$\to \frac{d^2x^k}{d\lambda^2} + \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij} \right) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = 0$$

g^{kl} = inverse metric

$$g^{kl} g_{li} = \delta^k_i = \mathbb{1}$$

Let $\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij} \right)$

→ Christoffel Symbols
or Connection Coefficients

Flat : $ds^2 = dx^2 + dy^2 \Rightarrow g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\boxed{\frac{d^2x^k}{d\lambda^2} + \Gamma^k_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = 0}$$

$\lambda \to a\lambda + b$

timelike paths: $ds^2 < 0$

Proper time: $d\tau^2 = -ds^2$

> 0 timelike paths

$$\tau = \int \sqrt{d\tau^2} = \int \sqrt{-ds^2} = \int \sqrt{-g_{\mu\nu}\, dx^\mu dx^\nu}$$

$$= \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$$

τ is the time measured by clocks moving along $x^\mu(\lambda)$.

Geodesics: $\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$

→ Maximize the proper time

Newtonian limit (static, weak)

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)[dx^2+dy^2+dz^2]$$

Φ = gravitational potl. ($\nabla^2\Phi = 4\pi G\rho$)

Geodesics: $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$

particles moving slowly: $\left|\frac{dx^i}{d\tau}\right| \ll \left|\frac{dx^0}{d\tau}\right|$

$\tau \approx t$

$\underline{\mu = i}$: $\frac{d^2x^i}{d\tau^2} + \Gamma^i_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$

$$\rightarrow \frac{d^2x^i}{dt^2} + \Gamma^i_{00}\left(\frac{dx^0}{dt}\right)^{\approx 1} = 0$$

$$\Gamma^i_{00} = \tfrac{1}{2}g^{i\lambda}(\partial_0 g_{0\lambda} + \partial_0 g_{\lambda 0} - \partial_\lambda g_{00})$$

$$= \tfrac{1}{2}g^{ii}(-\partial_i g_{00})$$

$$= \tfrac{1}{2}(1)(-\partial_i(-2\Phi)) = \partial_i\Phi$$

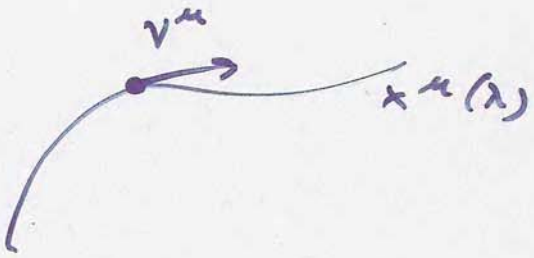
$$\rightarrow \boxed{\frac{d^2x^i}{dt^2} = -\partial_i\Phi}, \quad \vec{F} = m\vec{a} = -m\vec{\nabla}\Phi$$

Vectors

4-vectors $V^\mu = (V^0, V^1, V^2, V^3)$

- Tangent vector to a curve $x^\mu(\lambda)$:

$$V^\mu = \frac{dx^\mu}{d\lambda}$$



$g_{\mu\nu} V^\mu V^\nu < 0$ → timelike

- 4-velocity $U^\mu = \frac{dx^\mu}{d\tau} = \frac{d\lambda}{d\tau}\frac{dx^\mu}{d\lambda}$

$$g_{\mu\nu} U^\mu U^\nu = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \frac{(g_{\mu\nu}\, dx^\mu\, dx^\nu)}{d\tau^2}$$

$$= \frac{(-d\tau^2)}{d\tau^2} = -1$$

- 4-momentum $p^\mu = m U^\mu$

straight-line in Minkowski: $\gamma = \frac{1}{\sqrt{1-v^2}}$

$p^\mu = (m\gamma, m\gamma\vec{v})$ $\quad \vec{v}=0 \rightarrow \gamma = 1 \rightarrow E = m$

$m\gamma \rightarrow E$ $\quad m\gamma\vec{v} \rightarrow \vec{p}$

Tensors :

$g_{\mu\nu}$, $g^{\mu\nu}$, p^μ are all tensors :

collections of functions that transform correctly

e.g. $U^\mu = \frac{dx^\mu}{d\tau}$ $\qquad x^{\mu'} = x^{\mu'}(x^\mu)$

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu$$

note: $\frac{\partial x^\mu}{\partial x^{\mu'}}$ is the inverse of $\frac{\partial x^{\mu'}}{\partial x^\mu}$:

$$\frac{\partial x^{\mu'}}{\partial x^\nu} \frac{\partial x^\sigma}{\partial x^{\mu'}} = \delta^\sigma_\nu$$

$$U^{\mu'} = \frac{dx^{\mu'}}{d\tau} = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{dx^\mu}{d\tau} = \frac{\partial x^{\mu'}}{\partial x^\mu} U^\mu$$

Vectors: $U^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} U^{\mu}$

Metric: $ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = g_{\mu'\nu'} dx^{\mu'} dx$

divide by $dx^{\mu'} dx^{\nu'}$:

$$\boxed{g_{\mu'\nu'} = g_{\mu\nu} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}}}$$

Tensor transformation law:

$$S^{\alpha'\beta'\gamma'}{}_{\omega'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta'}}{\partial x^{\beta}} \frac{\partial x^{\gamma'}}{\partial x^{\gamma}} \frac{\partial x^{\omega}}{\partial x^{\omega'}} S^{\alpha\beta\gamma}{}_{\omega}$$

Index slot is important:

$$S^{\alpha\ \gamma}_{\sigma\ \ \omega} = g_{\sigma\beta} S^{\alpha\beta\gamma}{}_{\omega}$$

Contract indices!

$$S^{\alpha\beta} = S^{\alpha\beta\lambda}{}_{\lambda}$$

$$\tilde{S}^{\alpha\gamma} = S^{\alpha\lambda\gamma}{}_{\lambda}$$

~~$S^{\lambda\lambda\gamma}{}_{\omega}$~~

Inner product: $g_{\mu\nu} A^\mu B^\nu$

"Lowering indices": $A_\nu = g_{\mu\nu} A^\mu$

$$g_{\mu\nu} A^\mu B^\nu = A_\nu B^\nu$$

"Raising": $g^{\rho\sigma} A_\sigma = A^\rho$

$$g_{\mu\nu} V^\nu = g_{\mu\nu} (g^{\nu\sigma} V_\sigma)$$

$$[g_{\mu\nu} g^{\nu\sigma} = \delta_\mu^\sigma)$$

$$g_{\mu\nu} V^\nu = \delta_\mu^\sigma V_\sigma = V_\mu$$ ✓

Symmetric: $S^{\alpha\beta\gamma}{}_{\omega} = S^{\beta\alpha\gamma}{}_{\omega}$

Antisymmetric: $S^{\alpha\beta\gamma}{}_{\omega} = -S^{\beta\alpha\gamma}{}_{\omega}$

e.g. $g_{\mu\nu} = g_{\nu\mu}$ is symmetric

$F_{\mu\nu} = -F_{\nu\mu}$ is anti-

$$= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

Add: $A_{\alpha\beta} + B_{\alpha\beta} = C_{\alpha\beta}$ $\quad (k,l)+(k,l) = (k,l)$

(k,l): $T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$

Multiply: $A^{\alpha} B^{\beta}{}_{\gamma} = C^{\alpha\beta}{}_{\gamma}$

$(k,l) \otimes (m,n) = (k+m, l+n)$

$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$ $\quad T_{\alpha\beta} = T_{(\alpha\beta)} + T_{[\alpha\beta]}$

Minkowski Spacetime

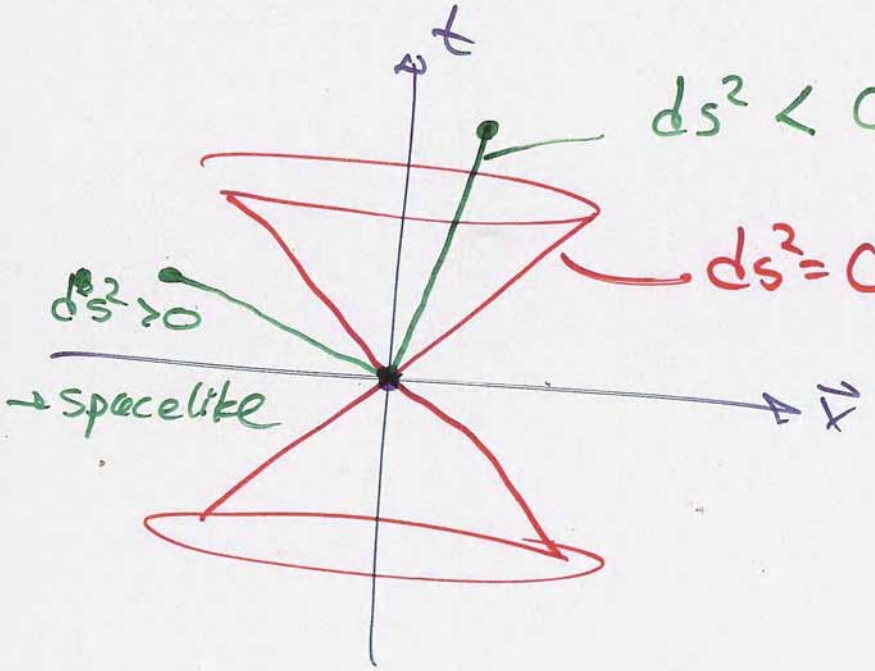
- $x^i \longrightarrow x^\mu = \{x^0, x^1, x^2, x^3\}$
  $= \{t, x, y, z\}$

- minus signs

Metric: $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$
$$= \eta_{\mu\nu}\, dx^\mu\, dx^\nu$$

$ds^2 = 0 \rightarrow$ lightlike (null)



Non - flat metrics

- Expanding universe:

$$ds^2 = -dt^2 + a^2(t)\underbrace{[dx^2 + dy^2 + dz^2]}_{\hookrightarrow \text{ or } S^3 \text{ or } \mathbb{H}^3}$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^2 & & \\ & & a^2 & \\ & & & a^2 \end{pmatrix}$$

- Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

Birkhoff's Theorem:

Schwarzschild is the unique spherically symmetric solution to Einstein's Eq. in vacuum.

Non-Tensors

① Partial derivatives

Gradient: $\partial_\mu \phi = \frac{\partial \phi}{\partial x^\mu}$

$$\partial_{\mu'} \phi = \frac{\partial \phi}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial \phi}{\partial x^\mu} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \phi$$

(0,1) tensor = dual vector

But $\partial_\mu V^\nu = \frac{\partial V^\nu}{\partial x^\mu}$

$$\partial_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu \right)$$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu}$$

q ☹

② Christoffel symbols

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\lambda} (\partial_\rho g_{\sigma\lambda} + \partial_\sigma g_{\lambda\rho} - \partial_\lambda g_{\rho\sigma})$$

$$\Gamma^{\mu'}_{\rho'\sigma'} = \sim\sim\sim$$

$$= \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial x^\sigma}{\partial x^{\sigma'}} \Gamma^\mu_{\rho\sigma} - \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial x^\sigma}{\partial x^{\sigma'}} \frac{\partial^2 x^{\mu'}}{\partial x^\rho \partial x^\sigma}$$

(3) Determinant of $g_{\mu\nu}$

$$g = |g_{\mu\nu}| = \text{Det}\, g_{\mu\nu}$$

$x^\mu \to x^{\mu'}$: $$g' = \left|\frac{\partial x^\mu}{\partial x^{\mu'}}\right|^2 g$$

$\to$ tensor density

(4) Volume element $d^n x = dx^0 dx^1 \ldots dx^{n-}$

$$d^n x' = \left|\frac{\partial x^{\mu'}}{\partial x^\mu}\right| d^n x$$

Tensorial volume element

$$\boxed{\sqrt{-g}\, d^n x = \sqrt{-g'}\, d^n x'}$$

$$X^{\alpha\beta}{}_\gamma = Y^{\alpha\beta}{}_\gamma$$

$$\to X^{\alpha'\beta'}{}_{\gamma'} = Y^{\alpha'\beta'}{}_{\gamma'}$$

$$\partial_\mu j^\mu = 0$$

Covariant Derivatives

Recall: $$\partial_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu}$$

$$\boxed{\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda}$$

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu$$

Dual vectors:

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda$$

$$\nabla_\mu S^\alpha{}_\beta = \partial_\mu S^\alpha{}_\beta + \Gamma^\alpha_{\mu\lambda} S^\lambda{}_\beta - \Gamma^\lambda_{\mu\beta} S^\alpha{}_\lambda$$

Metric Compatibility:

$$\nabla_\mu g_{\rho\sigma} = 0 = \nabla_\mu g^{\rho\sigma}$$

$$\nabla_\mu (AB) = (\nabla_\mu A) B + A (\nabla_\mu B)$$

Gauge Theory

GR	E + M
V^μ	ψ
$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu$	$\psi' = e^{i\alpha} \psi$
$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$	$D_\mu \psi = \partial_\mu \psi - ie A_\mu \psi$
	$A_\mu \to A_\mu + \partial_\mu \lambda$
	$D_\mu \psi \to e^{i\alpha} D_\mu \psi$

Field strength : $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

$$[D_\mu, D_\nu]\psi = D_\mu D_\nu \psi - D_\nu D_\mu \psi$$

$$= F_{\mu\nu} \psi$$

Curvature Tensor

$$[\nabla_\mu, \nabla_\nu] V^\rho = \nabla_\mu(\nabla_\nu V^\rho) - \nabla_\nu(\nabla_\mu V^\rho)$$

$$= R^\rho{}_{\sigma\mu\nu} V^\sigma$$

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\sigma\mu}$$

↳ Riemann Curvature Tensor

Properties: $R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^\lambda{}_{\sigma\mu\nu}$$

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$$

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$$

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$$

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$$

" Bianchi Identity "

$$\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho} = 0$$

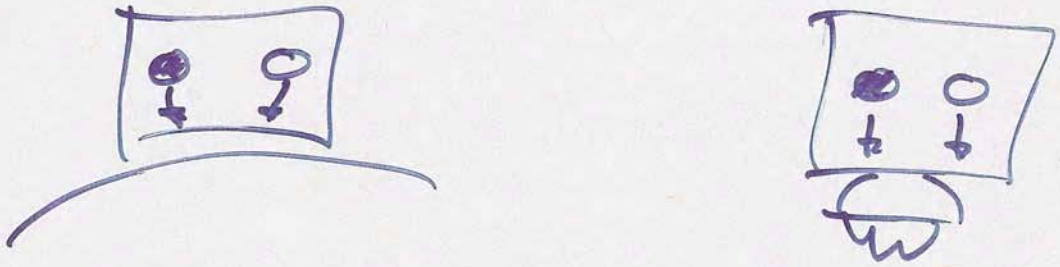
Contractions of $R^{\rho}{}_{\sigma\mu\nu}$:

Ricci tensor: $R_{\sigma\nu} = R^{\lambda}{}_{\sigma\lambda\nu}$

Curvature scalar: $R = g^{\sigma\nu} R_{\sigma\nu}$

Bianchi Identity → $\nabla^{\mu}(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = 0$

Physics? $R^{\rho}{}_{\sigma\mu\nu}$ → tidal forces



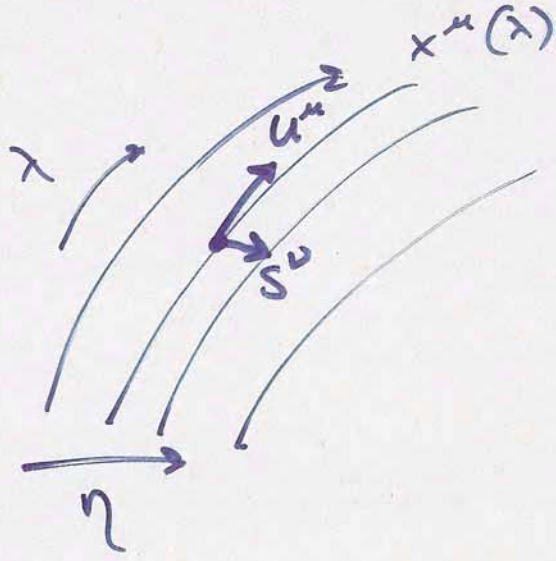
Geodesic Deviation

$U^{\mu} = \frac{dx^{\mu}}{d\lambda}$ @ η = const.

$S^{\nu} = \frac{dx^{\nu}}{d\eta}$ @ λ = const

$$\frac{D^2}{d\lambda^2} S^{\rho} = R^{\rho}{}_{\sigma\mu\nu} U^{\sigma} U^{\mu} S^{\nu}$$

$$\frac{D}{d\lambda} = \frac{dx^{\alpha}}{d\lambda} \nabla_{\alpha}$$



Einstein's Equation

Vary an action S_E wrt $g_{\mu\nu}(x)$

$$S = \int d^n x \sqrt{-g}\, \mathcal{L} = \int d^4 x \sqrt{-g}\, R$$

$$g^{\mu\nu} g_{\mu\nu} = 4$$

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \qquad R = g^{\mu\nu} R_{\mu\nu}$$

$$\delta S = \int d^n x \sqrt{-g} \left(R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu}$$

$$\Rightarrow \boxed{R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 0}$$

in vacuum

$$\nabla^2 \Phi = 0$$

Contract: $R - \tfrac{1}{2} R(4) = -R = 0$

$$\to \boxed{R_{\mu\nu} = 0}$$

Check:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2GM}{r}\right) & & & \\ & +\left(1 - \frac{2GM}{r}\right)^{-1} & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix}$$

Matter

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} + \mathcal{L}_M \right]$$

$S \to S + \delta S$ as $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$

$$\frac{1}{16\pi G}\left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right] + \frac{1}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}} = 0$$

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \rightarrow \text{Energy-Momentum Tensor} = T_{\mu\nu}$$

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G\, T_{\mu\nu}}$$

$$\nabla^\mu \left(R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}\right) = 0 \longrightarrow \nabla^\mu T_{\mu\nu} = 0$$

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{L}_\phi = -\frac{1}{2} \partial_\mu \phi \, \partial_\nu \phi \, g^{\mu\nu} - V(\phi)$$

Cosmology

Flat RW: $ds^2 = -dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right]$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^2 & & \\ & & a^2 & \\ & & & a^2 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^{-2} & & \\ & & a^{-2} & \\ & & & a^{-2} \end{pmatrix}$$

$g_{00} = -1, \; g_{ij} = a^2 \delta_{ij} \qquad g^{00} = -1, \; g^{ij} = a^{-2}\delta^{ij}$

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right)$$

$$\Gamma^{0}_{ij} = \frac{1}{2} g^{0\lambda}\left(\partial_i g_{j\lambda} + \partial_j g_{\lambda i} - \partial_\lambda g_{ij}\right)$$

$$= \frac{1}{2} g^{00}\left(-\partial_0 g_{ij}\right)$$

$$= \frac{1}{2}(-1)\left(-\partial_0 a^2 \delta_{ij}\right)$$

$$= \frac{1}{2}(-1)(-2)\dot{a}a\,\delta_{ij}$$

$$\rightarrow \quad \Gamma^{0}_{ij} = \dot{a}a\,\delta_{ij}$$

$$\Gamma^{i}_{j0} = \frac{\dot{a}}{a}\delta^{i}_{j}$$

all others $= 0$

Perfect Fluids

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}$$

ρ = energy density

p = pressure

U^μ = 4-velocity of fluid

Choose coords. @ point P s.t.

$$g_{\mu\nu}(P) = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$$

And $U^\mu = (1, 0, 0, 0)$

$$T_{\mu\nu} = \begin{pmatrix} \rho & & \\ & p\,\delta_{ij} & \end{pmatrix}$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^2 & & \\ & & a^2 & \\ & & & a^2 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^{-2} & & \\ & & a^{-2} & \\ & & & a^{-2} \end{pmatrix}$$

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p\, g^{\mu\nu}$$

$$u^\mu = (1, 0, 0, 0)$$

check: $g_{\mu\nu} u^\mu u^\nu = -1$

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & a^{-2} p & & \\ & & a^{-2} p & \\ & & & a^{-2} p \end{pmatrix}$$

$\delta^{ij} \delta_{ij} = 3$

$$\nabla_\mu T^{\mu\nu} = 0$$

<u>$\nu = 0$</u>: $\nabla_\mu T^{\mu 0} = 0$

$$= \partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu\lambda} T^{\lambda 0} + \Gamma^0_{\mu\lambda} T^{\mu\lambda}$$

$$= \partial_0 T^{00} + \sum_i \Gamma^i_{i0} T^{00} + \sum_i \Gamma^0_{ii} T^{ii}$$

$$= \partial_0 (\rho) + 3 \frac{\dot{a}}{a} \rho + 3 \frac{\dot{a}}{a} p = 0$$

$$\boxed{\dot{\rho} = -3 \frac{\dot{a}}{a} (\rho + p)}$$

$$\xrightarrow{p = w\rho} \boxed{\rho \propto a^{-3(1+w)}}$$

$$ds^2 = -dt^2 + a^2(t)\left(d\vec{x}^2\right)$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$R_{\sigma\upsilon} = R^{\lambda}{}_{\sigma\lambda\upsilon} = \partial_\lambda \Gamma^\lambda_{\sigma\upsilon} - \partial_\upsilon \Gamma^\lambda_{\sigma\lambda} + \Gamma^\lambda_{\lambda\omega}\Gamma^\omega_{\sigma\upsilon} - \Gamma^\lambda_{\omega\upsilon}\Gamma^\omega_{\sigma\lambda}$$

$$R_{00} = \partial_\lambda \Gamma^\lambda_{00} - \partial_0 \Gamma^\lambda_{0\lambda} + \Gamma^\lambda_{\lambda\omega}\Gamma^\omega_{00} - \Gamma^\lambda_{0\omega}\Gamma^\omega_{0\lambda}$$

$$= -\partial_0\left(3\frac{\dot{a}}{a}\right) + 0 - \left(\frac{\dot{a}}{a}\right)^2 \delta^i_j \delta^j_i$$

$$= -3\,\partial_0\left(\frac{\dot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)^2$$

$$= -3\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right] - 3\left(\frac{\dot{a}}{a}\right)^2$$

$$= -3\frac{\ddot{a}}{a} \quad \checkmark$$

$$\frac{\dot{a}}{a} = H$$

$$R_{ij} = \left(\ddot{a}a + 2\dot{a}^2\right)\delta_{ij}$$

$$\rho \propto a^{-n}$$

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \rightarrow a \propto t^{2/n}$$

$$R_{00} - \frac{1}{2} R g_{00} = 3\frac{\dot{a}^2}{a^2} = 8\pi G T_{00}$$

$$\Rightarrow \boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho}$$

Friedmann Equation