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Practical Twistor Spinoffs: On-Shell Tree and Loop Recursion Relations

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I briefly review how on-shell recursion relations, whose development was stimulated by recent twistor-space approaches, have been applied to compute tree and one-loop amplitudes in QCD.

1. INTRODUCTION

Much of the physics at a linear collider is precision physics. Theoretical predictions should match the anticipated experimental precision. In the case of QCD corrections to cross sections with a large number of final-state particles, one of the limitations is the lack of availability of one-loop QCD amplitudes. As just one example, next-to-leading order (NLO) QCD predictions for the production of four jets in e^+e^- annihilation exist [1], based in part on the one-loop amplitudes for a virtual photon or Z to decay into four partons [2]. However, the corresponding five-jet predictions cannot be computed at NLO until the one-loop amplitudes for $e^+e^- \to$ five partons are known.

Computing these amplitudes directly via Feynman diagrams would lead to very unwieldy expressions, because the amplitudes depend on a large number of kinematic variables. Individual Feynman diagrams lead to large sets of cumbersome loop integrals, and each diagram contains gauge-dependent pieces, which only cancel out in the full sum. Fortunately, there has been a great deal of recent progress, inspired by Witten's topological string theory in twistor space [3], towards understanding the structure of tree and loop amplitudes in QCD and related (supersymmetric) theories. This new understanding promises to make such computations considerably simpler, and to result in more compact, numerically stable representations of the final results. It has already accomplished this goal for many types of tree amplitudes, and for certain simple sets of loop amplitudes.

Essentially, the new methods allow the systematic construction of tree amplitudes from their basic analytic properties — poles in kinematic regions where the amplitude factorizes onto lower-point amplitudes [4, 5]. At loop level, the basic analytic properties include cuts as well as poles. Cuts can be computed using the unitarity method, making use of simple representations available for the tree-level helicity amplitudes on either side of the cut [6–8]. For loop amplitudes without cuts, it has recently been shown how to utilize the pole information recursively [9, 10], generalizing the tree level method [5]. The remaining task is to merge the cut information with the pole information, for the bulk of the loop amplitudes that do have cuts. Very recently, advances have been made in this direction [11].

The purpose of this talk is to briefly review some aspects of the recent developments, but there is no space to do justice to all the work in this direction. For a more complete picture of the recent twistor-inspired work at tree level and for one-loop amplitudes in supersymmetric theories, including also the connections to twistor space and twistor string theory, I refer the reader to ref. [12]. Here I describe on-shell recursion relations for tree amplitudes [4, 5], and give a simple application to a six-gluon amplitude in some detail. Then I very briefly discuss how the same types of techniques can be extended to one loop [9–11].

2. ON-SHELL RECURSION RELATIONS FOR TREE AMPLITUDES

2.1. The relations

In order to show the effectiveness of the new techniques, I would like to launch immediately into the set of on-shell recursion relations found by Britto, Cachazo and Feng [4], and proven by these authors and Witten [5].

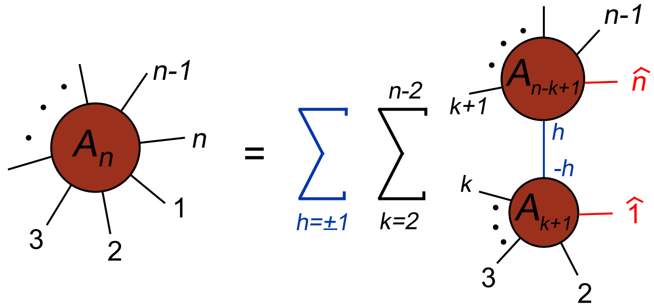

Figure 1: Diagrammatic representation of an on-shell recursion relation for tree amplitudes.

First, though, I quickly review [13] how color and spin information for multi-parton tree amplitudes is efficiently organized. For simplicity, consider the tree-level helicity amplitudes for n external gluons. (Amplitudes with massless external quarks can be handled in identical fashion [14].) Let all of the momenta k_i^μ be outgoing, so that the momentum conservation relation is $\sum_{i=1}^n k_i^\mu = 0$. For processes of physical interest with two incoming gluons, cross them into the initial state by letting those $k_i \to -k_i$. Similarly, label the gluon helicities as if all particles were outgoing. (The physical helicities of the two incoming gluons will therefore be the negative of this helicity label.)

The color information for the amplitude, carried by adjoint indices a_i for each gluon, is best disentangled using the trace-based color decomposition [13, 15–17]. The full amplitude $\mathcal{A}_n$ is a sum over color-ordered, partial amplitudes $A_n^{\rm tree}$, multiplied by traces of $SU(N_c)$ generator matrices in the fundamental representation (T^{a_i}),

$$\mathcal{A}_n^{\rm tree}(\{k_i,h_i,a_i\}) = g^{n-2} \sum_{\sigma\in S_n/Z_n} {\rm Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}})\, A_n^{\rm tree}(\sigma(1^{h_1},\ldots,n^{h_n}))\,, \tag{1}$$

where g is the strong coupling constant. The sum is over all non-cyclic permutations σ of n labels, S_n/Z_n. The labels of the $j^{\rm th}$ gluon are written as j^{h_j}, where $h_j = \pm 1$ is its (outgoing) helicity.

The on-shell recursion relations are for partial amplitudes with a standard ordering, $A_n^{\rm tree}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n})$. These amplitudes only have singular factorization behavior in a limited set of variables, involving squares of cyclicly-adjacent sums of momenta, of the form $K_{i,j}^2 \equiv (k_i + k_{i+1} + \cdots + k_{j-1} + k_j)^2$. Partial amplitudes with the other orderings required in eq. (1) are simply obtained by permuting the labels. The basic on-shell recursion relation for tree amplitudes reads [4, 5],

$$A_n^{\rm tree}(1,2,\ldots,n) = \sum_{h=\pm1} \sum_{k=2}^{n-2} A_{k+1}^{\rm tree}(\hat{1},2,\ldots,k,-\hat{K}_{1,k}^{-h}) \frac{i}{K_{1,k}^2} A_{n-k+1}^{\rm tree}(\hat{K}_{1,k}^{h}, k+1,\ldots,n-1,\hat{n}). \tag{2}$$

It is depicted diagrammatically in fig. 1. The amplitude is represented as a sum of products of lower-point amplitudes, evaluated on shell, but (as we shall see in a moment) for complex, shifted values of the momenta. The helicity labels of the n external gluons have been omitted, but they are the same on the left- and right-hand sides of eq. (2). For the relation to be valid, the helicities of gluons n and 1 can be $(h_n, h_1) = (-1, 1)$, $(1, 1)$, or $(-1, -1)$, but not $(1, -1)$. There are two sums. The first is over the helicity h of an intermediate gluon propagating (downward) between the two amplitudes. The second sum is over an integer k, which labels the different ways the set $\{1, 2, \ldots, n\}$ can be partitioned into two cyclicly-consecutive sets, each containing at least 3 elements, where labels 1 and n belong to different sets. A hat on top of a momentum label denotes that the corresponding momentum is *not* that of the original n-point amplitude, but is shifted to a different value.

To describe the shifted momenta, it is best to trade the four-momenta k_i^μ for spinor variables, namely the right- and left-handed, or $+$ and $-$ chirality, solutions to the Dirac equation, $u_\pm(k_i)$. A shorthand notation for the two-component (Weyl) versions of these spinors is $(\lambda_i)_\alpha \equiv u_+(k_i)$, $(\tilde\lambda_i)_{\dot\alpha} \equiv u_-(k_i)$. The form of the positive-energy projector for massless spinors is $u(k_i)\bar{u}(k_i) = \not{k}_i$, or in two-component notation,

$$k_i^\mu (\sigma_\mu)_{\alpha\dot\alpha} = (\not{k}_i)_{\alpha\dot\alpha} = (\lambda_i)_\alpha (\tilde\lambda_i)_{\dot\alpha}\,, \tag{3}$$

which shows that all momenta can be rewritten in terms of spinors. Amplitudes become functions of spinor inner-products [13], defined by

$$\langle j\, l\rangle = \varepsilon^{\alpha\beta}(\lambda_j)_\alpha(\lambda_l)_\beta = \bar{u}_-(k_j)u_+(k_l)\,, \qquad [j\, l] = \varepsilon^{\dot\alpha\dot\beta}(\tilde\lambda_j)_{\dot\alpha}(\tilde\lambda_l)_{\dot\beta} = \bar{u}_+(k_j)u_-(k_l)\,. \tag{4}$$

These products are antisymmetric, $\langle j\, i\rangle = -\langle i\, j\rangle$, $[j\, i] = -[i\, j]$, and are the square roots of the Lorentz products, up to a phase, because of the relation, $\langle j\, l\rangle\, [l\, j] = \frac{1}{2}\mathrm{tr}[(1-\gamma_5)\not{k}_j\not{k}_l] = 2k_j\cdot k_l = (k_j+k_l)^2$.

From eq. (3), k_i^μ is massless because of the antisymmetry of the spinor products,

$$k_i^2 = \varepsilon_{\beta\alpha}\varepsilon_{\dot\alpha\dot\beta}(\not{k}_i)^{\alpha\dot\alpha}(\not{k}_i)^{\dot\beta\beta} = \varepsilon_{\beta\alpha}\lambda_i^\alpha\lambda_i^\beta\varepsilon_{\dot\alpha\dot\beta}\tilde\lambda_i^{\dot\alpha}\tilde\lambda_i^{\dot\beta} = -\langle i\, i\rangle\,[i\, i] = 0\,. \tag{5}$$

It will continue to be massless even if one of the two spinors is shifted so that it is no longer the complex conjugate of the other spinor, for example

$$\hat{k}_i^\mu(\sigma_\mu)_{\alpha\dot\alpha} = (\hat{\not{k}}_i)_{\alpha\dot\alpha} = (\hat\lambda_i)_\alpha(\tilde\lambda_i)_{\dot\alpha}\,, \tag{6}$$

where $\hat\lambda_i$ is shifted away from λ_i.

The momentum shift in the k^{th} term in eq. (2) can now be described as,

$$\begin{aligned} \lambda_1 \;&\to\; \hat\lambda_1 \equiv \lambda_1 + z_k\lambda_n, \qquad && \tilde\lambda_1 \to \tilde\lambda_1, \\ \lambda_n \;&\to\; \lambda_n, && \tilde\lambda_n \to \hat{\tilde\lambda}_n \equiv \tilde\lambda_n - z_k\tilde\lambda_1\,, \end{aligned} \tag{7}$$

where

$$z_k = -\frac{K_{1,k}^2}{\langle n^-|\not{K}_{1,k}|1^-\rangle}\,. \tag{8}$$

This shift keeps $\hat{\not{k}}_1 = (\lambda_1 + z_k\lambda_n)\tilde\lambda_1$ and $\hat{\not{k}}_n = \lambda_n(\tilde\lambda_n - z_k\tilde\lambda_1)$ massless, as discussed above. It preserves overall momentum conservation, because $\hat{\not{k}}_1 + \hat{\not{k}}_n = \lambda_1\tilde\lambda_1 + \lambda_n\hat{\tilde\lambda}_n = \not{k}_1 + \not{k}_n$. And the intermediate gluon momentum, defined by $\hat{\not{K}}_{1,k} = \not{K}_{1,k} + z_k\lambda_n\tilde\lambda_1$, is also massless (on shell), because

$$\hat{K}_{1,k}^2 = (\not{K}_{1,k} + z_k\lambda_n\tilde\lambda_1)^2 = K_{1,k}^2 + z_k\langle n^-|\not{K}_{1,k}|1^-\rangle = 0\,. \tag{9}$$

The derivation of eq. (2) is very simple [5]. The momentum shift (7) is considered for an arbitrary complex number z, instead of the discrete values z_k in eq. (8). This shift defines an analytic function of z, $A_n^{\mathrm{tree}}(z)$. It has poles in z whenever a collection of the shifted momenta, corresponding to an intermediate state, can go on shell. For every allowed partition of $\{1,2,\ldots,n\}$ into $\{1,2,\ldots,k\}\cup\{k+1,\ldots,n-1,n\}$, there is a unique value of z that accomplishes this, z_k, because $\hat{K}_{1,k}^2(z_k) = 0$ according to eq. (9). The desired amplitude is the value of $A_n^{\mathrm{tree}}(z)$ at $z=0$. Provided that $A_n^{\mathrm{tree}}(z)\to 0$ as $z\to\infty$, this value at $z=0$ is determined by Cauchy's theorem in terms of the residues of $A_n^{\mathrm{tree}}(z)$ at $z=z_k$. Using general factorization properties of tree amplitudes, the k^{th} residue evaluates to the product found in the k^{th} term in eq. (2). The vanishing of $A_n^{\mathrm{tree}}(z)$ as $z\to\infty$ can be established directly from Feynman diagrams for $(h_n,h_1) = (-1,1)$ [5]. For the other two valid cases it can be obtained using a representation of the amplitude via "MHV rules" [18], or by a recursive argument [19].

Off-shell recursive approaches to summing Feynman diagrams have a long history [20, 21]. The virtue of the on-shell recursion relations is that the intermediate, lower-point quantities are not auxiliary, gauge-dependent quantities, as in the off-shell case; they are precisely the desired physical, on-shell scattering amplitudes, just with fewer partons.

No knowledge of twistor space is needed to implement eq. (2). Heuristically, its derivation is related to twistor space in that spinors, not vectors, play the fundamental role. (Twistor space is obtained by Fourier transforming the $\tilde\lambda_i$, while leaving the λ_i alone.) The actual motivation for eq. (2) was provided by a detour through one-loop amplitudes in supersymmetric theories, and their twistor-space structure, which is too lengthy to describe here.

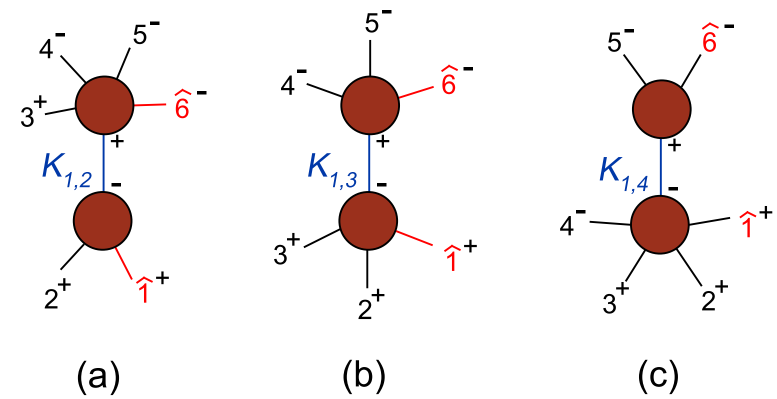

Figure 2: On-shell recursive diagrams for $A_6^{\rm tree}(1^+,2^+,3^+,4^-,5^-,6^-)$.

2.2. A simple application

Let us now work through a simple application of eq. (2) [4]. The n-gluon amplitudes for which all gluons, or all but one, have the same helicity, vanish identically (as may be proved by a supersymmetry Ward identity [22]),

$$A_n^{\rm tree}(1^\pm,2^+,3^+,\ldots,n^+) = 0\,. \tag{10}$$

The first sequence of nonvanishing tree amplitudes has two gluons (j and l) with opposite helicity to the rest, and is called maximally helicity-violating (MHV). These amplitudes have the exceedingly simple expression [16, 17, 23]

$$A_n^{{\rm MHV}\,,jl} \equiv A_n^{\rm tree}(1^+,2^+,\ldots,j^-,\ldots,l^-,\ldots,n^+) = i\frac{\langle j\,l\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,. \tag{11}$$

One can use parity, which exchanges spinor products $\langle i\,j\rangle \leftrightarrow [i\,j]$, to reverse all gluon helicities in an amplitude. Thus eqs. (10) and (11) exhaust all of the five-gluon tree amplitudes. For six gluons, there are three cyclicly-inequivalent "next-to-maximally helicity-violating" (NMHV) amplitudes: $A_6^{\rm tree}(1^+,2^+,3^+,4^-,5^-,6^-)$, $A_6^{\rm tree}(1^+,2^+,3^-,4^+,5^-,6^-)$, and $A_6^{\rm tree}(1^+,2^-,3^+,4^-,5^+,6^-)$. The last of these amplitudes can be related to the first two by group theory (a "dual Ward identity" or "$U(1)$ decoupling identity" [13]). Here we will compute the first one using eq. (2). Instead of 220 Feynman diagrams (including all color-orderings), there are just three potential on-shell recursive diagrams, shown in fig. 2. Diagrams of the form of fig. 2(a) and fig. 2(c), but with a reversed helicity assignment to the intermediate gluon, vanish because $A_3^{\rm tree}(+,+,+) = A_3^{\rm tree}(-,-,-) = 0$, and have been omitted. Figure 2(b), and the corresponding diagram with a reversed intermediate helicity, both vanish using eq. (10). Finally, diagram (c) is related to diagram (a) by the "flip" symmetry $(1\leftrightarrow 6, 2\leftrightarrow 5, 3\leftrightarrow 4)$ (plus spinor conjugation).

Remarkably, only one diagram, fig. 2(a), is required to compute the helicity amplitude $A_6^{\rm tree}(1^+,2^+,3^+,4^-,5^-,6^-)$. Its value is given by the product of two shifted MHV amplitudes, each parity-conjugated with respect to eq. (11),

$$\begin{aligned} D^{(a)} &= A_3^{\rm tree}(\hat{1}^+,2^+,-\hat{K}_{1,2}^-)\frac{i}{K_{1,2}^2}A_5^{\rm tree}(\hat{K}_{1,2}^+,3^+,4^-,5^-,\hat{6}^-) \\ &= \frac{-i}{s_{12}}\frac{[\hat{1}\,2]^3}{[2\,\hat{K}]\;[\hat{K}\,\hat{1}]}\frac{[\hat{K}\,3]^3}{[3\,4]\,[4\,5]\;[5\,\hat{6}]\;[\hat{6}\,\hat{K}]} = \frac{-i}{s_{12}}\frac{[1\,2]^3}{(\,[2\,\hat{K}]\;\langle\hat{K}\,6\rangle)(\,\langle 6\,\hat{K}\rangle\;[\hat{K}\,1])}\frac{(\,\langle 6\,\hat{K}\rangle\;[\hat{K}\,3])^3}{[3\,4]\,[4\,5]\;[5\,\hat{6}]\,(\,[\hat{6}\,\hat{K}]\;\langle\hat{K}\,6\rangle)}\,, \end{aligned} \tag{12}$$

where $\hat{K} = \hat{K}_{1,2}$. Multiplying top and bottom by factors of $\langle 6\,\hat{K}\rangle = -\langle\hat{K}\,6\rangle$ makes subsequent steps simpler. Now,

$$\hat{\not{K}} = \not{k}_1 + \not{k}_2 - \frac{\langle 1\,2\rangle\,[2\,1]}{\langle 6\,2\rangle\,[2\,1]}\lambda_6\tilde{\lambda}_1\,, \qquad \hat{\tilde{\lambda}}_6 = \tilde{\lambda}_6 + \frac{\langle 1\,2\rangle\,[2\,1]}{\langle 6\,2\rangle\,[2\,1]}\lambda_6\,, \tag{13}$$

so we have (for any a),

$$\langle 6\,\hat{K}\rangle\;[\hat{K}\,a] \;=\; \langle 6^-|\hat{\not{K}}|a^-\rangle = \langle 6\,1\rangle\,[1\,a] + \langle 6\,2\rangle\,[2\,a] = \left\langle 6^-\right|(1+2)\left|a^-\right\rangle\,, \tag{14}$$

$$[5\,\hat{6}] \;=\; [5\,6] + \frac{\langle 1\,2\rangle\,[2\,1]}{\langle 6\,2\rangle\,[2\,1]}\,[5\,1] = \frac{\left\langle 5^+\right|(6+1)\left|2^+\right\rangle}{\langle 6\,2\rangle} = \frac{\left\langle 2^-\right|(6+1)\left|5^-\right\rangle}{\langle 6\,2\rangle}\,, \tag{15}$$

$$[\hat{6}\,\hat{K}]\;\langle \hat{K}\,6\rangle \;=\; \left\langle 6^+\right|(1+2)\left|6^+\right\rangle + \langle 1\,2\rangle\,[2\,1] = s_{612}\,, \tag{16}$$

where $s_{612} = (k_6+k_1+k_2)^2$. Inserting these expressions into eq. (12) yields the one-term expression,

$$D^{(a)} = i\,\frac{\left\langle 6^-\right|(1+2)\left|3^-\right\rangle^3}{\langle 6\,1\rangle\,\langle 1\,2\rangle\,[3\,4]\,[4\,5]\,s_{612}\left\langle 2^-\right|(6+1)\left|5^-\right\rangle}\,. \tag{17}$$

Adding the image of this term under $(1\leftrightarrow 6, 2\leftrightarrow 5, 3\leftrightarrow 4)$ (plus spinor conjugation), the full amplitude is,

$$A_6^{\rm tree}(1^+,2^+,3^+,4^-,5^-,6^-) = \frac{i\,\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\,\langle 1\,2\rangle\,[3\,4]\,[4\,5]\,s_{612}\langle 2^-|(6+1)|5^-\rangle} + \frac{i\,\langle 4^-|(5+6)|1^-\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,[5\,6]\,[6\,1]\,s_{561}\langle 2^-|(6+1)|5^-\rangle}\,. \tag{18}$$

3. ON-SHELL RECURSION RELATIONS FOR ONE-LOOP AMPLITUDES

Most one-loop amplitudes contain cuts. However, there are three infinite sequences of QCD amplitudes that do not, essentially because the corresponding tree amplitudes vanish. Two of these sequences are given in eq. (10). The third sequence contains a pair of massless external quarks, and the remaining $(n-2)$ gluons all contain the same helicity. Because these purely-rational amplitudes are very "tree-like", it is possible to write down on-shell recursion relations for all three sequences, and find compact solutions for them [9, 10]. There are some subtleties not encountered in the tree-level case. In particular, shifted one-loop amplitudes depending on a complex parameter z can contain double poles at some values of z. The residue underneath this double pole has been determined empirically.

The color decomposition of a one-loop n-gluon amplitude contains leading-in-N_c, single-trace terms just like eq. (1), but with $A_n^{\rm tree}$ replaced by a one-loop partial amplitude $A_n^{(1)}$, and an additional overall factor of $g^2N_c/(4\pi)^2$. (There are also subleading-in-N_c, double-trace terms, which are given by sums over permutations of the $A_n^{(1)}$ [6].) The recursion relation obeyed by the set of one-loop n-gluon amplitudes with precisely one negative-helicity gluon is [9],

$$\begin{aligned}
A_n^{(1)}(1^-,2^+,\ldots,n^+) &= A_{n-1}^{(1)}(4^+,5^+,\ldots,n^+,\hat{1}^-,\hat{K}_{2,3}^+)\,\frac{i}{K_{2,3}^2}\,A_3^{\rm tree}(\hat{2}^+,3^+,-\hat{K}_{2,3}^-) \\
&+\sum_{j=4}^{n-1} A_{n-j+2}^{\rm tree}((j+1)^+,(j+2)^+,\ldots,n^+,\hat{1}^-,\hat{K}_{2,j}^-)\,\frac{i}{K_{2,j}^2}\,A_j^{(1)}(\hat{2}^+,3^+,\ldots,j^+,-\hat{K}_{2,j}^+) \\
&+ A_{n-1}^{\rm tree}(4^+,5^+,\ldots,n^+,\hat{1}^-,\hat{K}_{2,3}^-)\,\frac{i}{(K_{2,3}^2)^2}\,V_3^{(1)}(\hat{2}^+,3^+,-\hat{K}_{2,3}^+)\times\Big(1+K_{2,3}^2\,\mathcal{S}^{\rm tree}(\hat{1},\hat{K}_{2,3}^+,4)\,\mathcal{S}^{\rm tree}(3,-\hat{K}_{2,3}^-,\hat{2})\Big)\,,
\end{aligned} \tag{19}$$

where

$$\mathcal{S}^{\rm tree}(a,s^+,b) = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\,\langle s\,b\rangle}\,, \qquad \mathcal{S}^{\rm tree}(a,s^-,b) = -\frac{[a\,b]}{[a\,s]\,[s\,b]}\,. \tag{20}$$

The hatted momenta are evaluated using the shift $\tilde\lambda_1 \to \tilde\lambda_1 - z_j\tilde\lambda_2$, $\lambda_2\to\lambda_2+z_j\lambda_1$; for the $K_{2,j}$ channel,

$$z_j = -\frac{K_{2,j}^2}{\left\langle 1^-\right|\not{K}_{2,j}\left|2^-\right\rangle}\,. \tag{21}$$

The recursion relation (19), and a similar one for the case with a massless external quark pair, have compact solutions for all n [10]. The validity of the solutions can be verified using, in part, a QCD result and related QED results, obtained previously via off-shell recursive techniques [21]. Although these cut-free helicity amplitudes are very special, extensions of the same techniques allow one to recursively construct one-loop amplitudes with cuts, such as the very recently computed $A_6^{(1)}(1^-,2^-,3^+,4^+,5^+,6^+)$ [11], after determining the cuts via unitarity [6, 7].

In conclusion, there has been a lot of recent progress in using the analytic structure of tree and one-loop QCD amplitudes as a computational tool, codified in the form of on-shell recursion relations. Much of this progress was indirectly stimulated by twistor-space ideas [3]. The prospects for using these techniques to compute many new, phenomenologically relevant, one-loop, multi-parton amplitudes in the near future are quite good.

Acknowledgments

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