# Creation of a Brane World with Gauss-Bonnet Term

Koh-suke Aoyanagi
*Department of Physics, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan*
Kei-ichi Maeda
*Advanced Research Institute for Science and Engineering, Waseda University, Shinjuku, Tokyo 169-8555, Japan*
*Waseda Institute for Astrophysics, Waseda University, Shinjuku, Tokyo 169-8555, Japan*

Here we study the creation of a brane world using an instanton solution with Hartle-Hawking's no boundary approach. We analyze brane models with a Gauss-Bonnet term in a bulk spacetime. The curvature of 3-brane is assumed to be closed, flat, or open. We construct instanton solutions with branes for our models, and calculate the value of the actions to discuss the initial state of a brane universe.

## 1. INTRODUCTION

Although a big bang universe is very successful, it predicts the existence of an initial singularity. To resolve such a difficulty, we have to search for a new gravitation theory, such as quantum gravity. However, we could so far not find it. As a first step toward reaching it, we may consider a mini-super space and quantize the isotropic and homogeneous universe, which is the so-called quantum cosmology [1, 2]. Vilenkin claimed that the universe is created from nothing [1]. This approach is based on the picture that the universe is spontaneously nucleated in a de Sitter space. The mathematical description of this nucleation is analogous to a quantum tunneling through a potential barrier [3]. Another approach to quantum cosmology was developed by Hartle and Hawking [2], who proposed that the wave function of the universe is given by a path integral over non-singular compact Euclidean geometries, which is called a "no boundary" boundary condition. Under this condition, we consider the creation of a brane world. Then the wave function is expected to be proportional to $e^{-S_E}$, where $S_E$ is the Euclidean action.

When we discuss the early stage of the universe, however, a unified theory of fundamental interactions and particles will play a very important role. Among such unified theories, string/M theory is the most promising candidate, which is constructed in higher-dimensional spacetime. Based on such higher-dimensional theories, a new cosmological scenario has been proposed, that is, a brane cosmology. One of the most interesting approach was given by Randall and Sundrum [4, 5]. They considered a pure 5-dimensional (5D) Einstein gravity in a bulk only with a cosmological constant. In their second model [5], it was shown that four-dimensional gravity is recovered even in an infinite bulk spacetime.

If we believe such a higher-dimensional cosmological scenario, we have to invoke how such a universe is created. In particular, because a brane structure is highly inhomogeneous in a higher-dimensional bulk spacetime, we may wonder how such a brane universe is born and starts to evolve. As for creation of a brane universe, a some work has been so far done. Garriga and Sasaki first constructed an inflating brane instanton of the Randall-Sundrum model [6]. This instanton is obtained by gluing two spherical parts of $\mathrm{AdS}_5$. It was also considered that the creation using an instanton, inflation and a fluctuations during the de Sitter phase in the model containing the quantum correction term called a trace anomaly on the brane [7].

When we discuss such quantum effects on the brane, we may also have to include quantum effects in the bulk. In the higher dimensional theory, the higher curvature correction terms should be added to the Einstein-Hilbert action. These terms appear in the low energy effective action of string theory via quantum one-loop corrections. In fact the low energy effective actions of some string theories include $\mathcal{R}^{ABCD}\mathcal{R}_{ABCD}$ interactions, but this term gives rise to a ghost. In order to resolve this problem, a ghost-free Gauss-Bonnet combination was introduced [8]. Hence, the Gauss-bonnet term should be included when we discuss a brane universe with some quantum corrections.

Here we consider the creation of a brane universe using an instanton solution in the Einstein-Gauss-Bonnet theory [9]. We construct a brane instanton including a Gauss-Bonnet term.

## 2. ACTION AND EQUATIONS OF MOTION

The Euclidean action for the brane world with a Gauss-Bonnet term in 5D spacetime is described by two parts: one in a bulk spacetime ($\mathcal{M}$) and the other in brane boundary hypersurfaces ($\partial\mathcal{M} = \sum_i \partial\mathcal{M}_i$), i.e.,

$$S_E = S_E^{\rm bulk} + S_E^{\rm brane}\,. \tag{1}$$

The bulk action is given by

$$S_E^{\rm bulk} = -\frac{1}{2\kappa_5^2}\int_{\mathcal{M}} dx^5\sqrt{g}\left[\mathcal{R} - 2\Lambda + \alpha\mathcal{L}_{GB}\right], \tag{2}$$

where $\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{AB}\mathcal{R}^{AB} + \mathcal{R}_{ABCD}\mathcal{R}^{ABCD}$, $\kappa_5^2$ is the 5D gravitational constant, $\Lambda$ is a cosmological constant, $\mathcal{R}, \mathcal{R}_{AB}$, and $\mathcal{R}_{ABCD}$ are the 5D scalar

curvature, the Ricci tensor and the Riemann tensor, respectively, and $\alpha$ is a constant, which is related to a string coupling constant. The induced 4D metric $h_{\mu\nu}$ on a 3-brane is defined by $h_{AB} = g_{AB} - n_A n_B$, where $n_A$ is the spacelike unit vector field normal to the brane hypersurface. The action of the branes is given by the following form:

$$S_E^{\rm brane} = -\sum_i \int_{\partial\mathcal{M}_i} d^4x\sqrt{h}\left[\frac{1}{\kappa_5^2}L_{\rm surface}(\partial\mathcal{M}_i) - \lambda_i\right], \quad (3)$$

where $L_{\rm surface} = K + 2\alpha(J - 2G^{\rho\sigma}K_{\rho\sigma})$ is a surface term of the 5D gravitational action [10]. $\lambda_i$ is a tension on the $i$-th brane, $K_{\mu\nu}$ is the extrinsic curvature of a brane, $K = K^\mu_\mu$, and $G_{\mu\nu}$ is the Einstein tensor of the induced metric $h_{\mu\nu}$. $J$ is a trace of $J_{\mu\nu}$, which is given by some combination of the extrinsic curvature defined by

$$\begin{aligned} J_{\mu\nu} &= \frac{1}{3}\left(2KK_{\mu\rho}K^\rho_{\ \nu} + K_{\rho\sigma}K^{\rho\sigma}K_{\mu\nu}\right. \\ &\quad \left. -2K_{\mu\rho}K^{\rho\sigma}K_{\sigma\nu} - K^2K_{\mu\nu}\right). \end{aligned} \quad (4)$$

The total action ($S_E = S_E^{\rm bulk} + S_E^{\rm brane}$) gives the field equations as

$$\mathcal{G}_{AB} + \alpha\mathcal{H}_{AB} = -\Lambda g_{AB} - \sum_i \lambda_i g_{AB}\delta(\partial\mathcal{M}_i), \quad (5)$$

where

$$\mathcal{G}_{AB} = \mathcal{R}_{AB} - \frac{1}{2}g_{AB}\mathcal{R}, \quad (6)$$

and

$$\begin{aligned} \mathcal{H}_{AB} &= 2\left[\mathcal{R}\mathcal{R}_{AB} - 2\mathcal{R}_{AC}\mathcal{R}^C_{\ B} - 2\mathcal{R}^{CD}\mathcal{R}_{ACBD}\right. \\ &\quad \left. +\mathcal{R}_A^{\ CDE}\mathcal{R}_{BCDE}\right] - \frac{1}{2}g_{AB}\mathcal{L}_{GB}. \end{aligned} \quad (7)$$

Since we are looking for an instanton solution, we assume a highly symmetric Euclidean spacetime, whose metric is given by

$$ds_E^2 = dr^2 + b(r)^2\gamma_{\mu\nu}dx^\mu dx^\nu, \quad (8)$$

where $\gamma_{\mu\nu}$ is a 4D metric with maximal symmetry. This maximally symmetric 4D space is classified into three cases by the signature of curvature, i.e., $k$ =0 (zero), 1 (positive), or $-1$ (negative). It corresponds to the curvature sign of the Friedmann universe after creation. Since the Euclidian space must be compact when we discuss its creation, in the case of $k$=0 or $-1$, we have to make a space compact by identification. Then the flat spacetime is a 4D torus, and that with $k = -1$ has a more complicated topology. Although the spacetimes are compact, we shall call them "flat" and "open" for $k = 0$ and $-1$ as well as "closed" for $k = 1$.

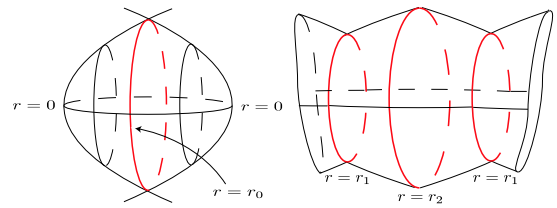


Figure 1: Brane instanton. The thick vertical circle at $r = r_i$ represents the four-sphere brane at which the two identical five-dimensional anti-de Sitter spaces are glued.

The equations of motion under the above ansatz are given by

$$\begin{aligned} 3\left(\frac{b''}{b} - \frac{k-b'^2}{b^2}\right) + 12\alpha\frac{(k-b'^2)}{b^3}b'' \\ = -\Lambda - \sum_i \bar{\lambda}_i\delta(r-r_i), \end{aligned} \quad (9)$$

$$6\frac{(k-b'^2)}{b^2}\left\{1 + 2\alpha\frac{(k-b'^2)}{b^2}\right\} = \Lambda, \quad (10)$$

where the prime denotes the derivative with respect to $r$ and $\bar{\lambda}_i = \kappa_5^2\lambda_i$.

By integrating the first equation for a small interval $(r_i-\epsilon, r_i+\epsilon)$ including a brane, we obtain the junction condition [11] at $r = r_i$ as

$$\left[\frac{b'}{b}\left\{3 - 4\alpha\left(\frac{b'^2}{b^2} - \frac{3k}{b^2}\right)\right\}\right]^\pm = -\bar{\lambda}_i\,, \quad (11)$$

where $[\ ]^\pm$ denotes

$$[A]^\pm = A(r_i+\epsilon) - A(r_i-\epsilon) \equiv A_+ - A_-. \quad (12)$$

With the ansatz of $Z_2$ symmetry, which gives the relation of $[A]^\pm = 2A_+ = -2A_-$, we obtain

$$\frac{b'}{b}\left\{3 - 4\alpha\left(\frac{b'^2}{b^2} - \frac{3k}{b^2}\right)\right\} = \mp\frac{\bar{\lambda}_i}{2}. \quad (13)$$

Here the upper (lower) sign is applied at $r = r_2$ (at $r = r_1$) for a two-brane model. For a single brane model, we apply the upper case at $r = r_0$. Throughout this paper, we use the notation $r_i (i = 0, 1, 2)$, where $i$ means the number of a brane. For a two-brane model $i = 1$ and 2 stand for a negative and a positive tension brane, respectively. While for a single brane model, we use $i = 0$ to stand for a brane (see Fig. 1).

## 3. INSTANTON SOLUTION

We first provide a solution of Eqs. (9) and (10) in a bulk. Eq. (10) gives the quadratic equation,

$$6X + 12\alpha X^2 = \Lambda, \quad (14)$$

where

$$X \equiv \frac{k - b'^2}{b^2}. \tag{15}$$

If $\alpha \neq 0$, Eq. (14) gives two solutions:

$$X = X_\pm \equiv \frac{-1 \pm \eta}{4\alpha}, \tag{16}$$

where $\eta = \sqrt{1 + 4\alpha\Lambda/3}$. Here we find two solutions; one is called a plus-branch and the other is a minus-branch. The limit of $\alpha \to 0$ exists only for the plus-branch solution.

In this paper we consider only a negative cosmological constant ($\Lambda < 0$). [1] Then we have the following constraint,

$$-\frac{3}{4\alpha} \leq \Lambda < 0\,, \tag{17}$$

which is required in order that $X_\pm$ be a real value. The range of $\eta$ is restricted as $0 \leq \eta < 1$ from the constraint (17). Under this condition, $X_\pm$ is always negative. We then introduce a typical length scale as

$$l_\pm \equiv (-X_\pm)^{-1/2} = \left[-\frac{3(1 \pm \eta)}{\Lambda}\right]^{1/2}. \tag{18}$$

Since $X' = 0$, we find $b'' = l_\pm^2 b$ from Eq. (15). This with Eq. (14) guarantees Eq. (9). Hence Eq. (16) gives a bulk solution.

In what follows, we discuss the instanton solutions for each value of $k$ in order.

## 3.1. de Sitter brane instanton ($k = 1$)

In the case of $k = 1$, the solution (16) is written by,

$$b(r) = l_\pm \sinh\left(\frac{r}{l_\pm}\right)\,, \tag{19}$$

which also satisfies Eq. (9) in a bulk. From this bulk solution, we construct an instanton solution by cutting the space at $r = r_i$ and gluing two copies of it on the surfaces of the corresponding point so that a compact Euclidean manifold (instanton) is obtained. At $r = r_i$, we impose the Israel's junction condition (13) with $k = 1$. For a single brane instanton, we impose the "no boundary boundary condition" at the origin [2]. For a two-brane model, we impose the junction condition at $r_1$ and $r_2$. As a result, the tension of $i$-th brane is determined by these junction conditions. Substituting (16) into (13) gives

$$\bar{\lambda}_i^{(\pm)} = (-1)^i \frac{2}{l_\pm}\left[(2 \pm \eta)\frac{\cosh(r_i/l_\pm)}{\sinh(r_i/l_\pm)} + 2(1 \mp \eta)\left(\frac{\cosh(r_i/l_\pm)}{\sinh^3(r_i/l_\pm)}\right)\right] \tag{20}$$

When we take the limit of $\alpha \to 0$ in the plus-branch, we recover the Garriga-Sasaki instanton, i.e., $l_\pm = l \equiv \sqrt{-6/\Lambda}$, $\eta = 1$ , and

$$\bar{\lambda}_i = (-1)^i \frac{6}{l} \coth\left(\frac{r_i}{l}\right)\,. \tag{21}$$

Here we note a "critical" tension. In a brane model with a Gauss-Bonnet term in a bulk, we find some contribution from the Gauss-Bonnet term in a four-dimensional cosmological constant. As a result, the fine-tuned value of the tension to find the 4D Minkowski brane, which we shall call a critical tension, is modified from the Randall-Sundrum's value. The fine-tuned value is given by [12] as

$$\alpha\bar{\lambda}_{\rm cr}^2 = 1 - 4\alpha\Lambda \mp \left(1 + \frac{4}{3}\alpha\Lambda\right)^3. \tag{22}$$

In our case, if we take the limit of $r \to \infty$, the brane approaches to 4D Euclidian flat space because the radius of the brane ($b(r)$) becomes infinitely large and the curvature of the brane ($S^4$ manifold) vanishes. At this limit, the tension of the brane (21) is

$$\bar{\lambda}_{i,\rm cr} = (-1)^i \frac{2}{l_\pm}(2 \pm \eta). \tag{23}$$

This value is consistent with the above generalized Randall-Sundrum tuning condition (22). Using this critical tension, the tension of a positive-tension brane is divided into two parts $\bar{\lambda} = \bar{\lambda}_{\rm cr} + \triangle\bar{\lambda}$. It turns out that $\triangle\bar{\lambda}$ is always positive because $\bar{\lambda}$ decreases monotonically with respect to $r$ for $0 \leq \eta < 1$ Hence, this brane has always a positive effective cosmological constant, that is, the de Sitter brane.

We calculate the action in order to discuss which state is most plausible when the brane universe is created. The total Euclidian action for this solution is calculated as

$$S_E = -\frac{8\pi^2}{\kappa_5^2} l_\pm^3 \left\{\left[(2 \mp \eta)\sinh\left(\frac{r_2}{l_\pm}\right)\cosh\left(\frac{r_2}{l_\pm}\right) + (2 \mp 3\eta)\left(\frac{r_2}{l_\pm}\right)\right] - [r_2 \to r_1]\right\}. \tag{24}$$

For a single brane instanton, the action is given by replacing $r_2$ and $r_1$ with $r_0$ and 0, respectively.

Those actions do not have any minimum value, and get small when the distance between two branes or

[1] In the case of $\Lambda = 0$, there also exist instanton solutions. For $k = 1$, the minus-branch solution is included in (16), and the plus-branch solution is also included in the limit of $X_+ \to 0$. This plus-branch solution is a 5D Milne universe. For $k = 0$ the solutions of both branches is included in (27). For $k = -1$ only the minus-branch solution exists and is included in (33).

the size of the brane becomes large. Although we can claim that the brane universe may be created as large as possible, we cannot predict the initial size.

The evolution of a brane after creation is given by analytic continuation of the Euclidean metric

$$ds_E^2 = dr^2 + l^2 \sinh^2(r/l_\pm)\left(d\chi^2 + \sin^2\chi d\Omega_{(3)}^2\right) \quad (25)$$

by the Wick rotation, where $d\Omega_{(3)}^2$ is the metric of the 3-sphere. It is done by substituting $\chi \to iHt + \pi/2$, which leads to

$$\begin{aligned} ds^2 &= dr^2 + (l_\pm H)^2 \sinh(r/l_\pm) \\ &\quad \times[-dt^2 + H^{-2}\cosh^2(Ht)d\Omega_{(3)}^2], \end{aligned} \quad (26)$$

where $H \equiv l_\pm \sinh(r_i/l_\pm)$ ($r_i = r_0$ or $r_2$) is the radius of a brane. After the creation of this spacetime, the universe inflates. If $\triangle\bar{\lambda}$ is given by some potential of a scalar field and will decrease to zero, inflation will end (see Section 4).

## 3.2. Flat brane instanton ($k = 0$)

In the case of a flat brane ($k = 0$), the solution of Eq. (15) is $b(r) = b_0 e^{\pm r/l_\pm}$, where $b_0$ is an integrating constant. Due to the $Z_2$ symmetry, we consider only the plus sign without loss of generality, i.e.,

$$b(r) = b_0 e^{r/l_\pm}\,. \quad (27)$$

We can construct an instanton solution in the same way as the previous case. We can impose the junction conditions at the brane boundaries ($r = r_1$ and $r_2$, or $r_0$). This solution, however, does not satisfy the no-boundary boundary condition, because $b(r)$ does not vanish at any point $r$. Thus we cannot construct a single brane instanton solution. Here we consider only a two-brane model.

The tension of $i$-th brane is determined by the junction condition (13) as

$$\frac{b'}{b}\left(3 - 4\alpha\frac{b'^2}{b^2}\right) = \mp\frac{\bar{\lambda}_i}{2}. \quad (28)$$

Substituting Eq. (27) into Eq. (28),

$$\bar{\lambda}_i^{(\pm)} = (-1)^i \frac{2}{l_\pm}(2 \pm \eta)\,. \quad (29)$$

This tension is independent of the position of a brane and is the same as the critical tension (23).

As for the Euclidean action of this solution, we find

$$S_E = -\frac{2V_4^{(0)} b_0^4 (2 \mp 3\eta)}{\kappa_5^2\, l_\pm}\left[e^{4r_2/l_\pm} - e^{4r_1/l_\pm}\right]. \quad (30)$$

Here $V_4^{(0)}$ is the volume of a 4D torus. This action does also not have any minimum value with respect to the distance between two branes. Then we cannot predict the initial size of the universe. Note that we can obtain a single-brane RS II model in the limit of $r_1 \to -\infty$. In that case, the total action $S_E$ is still finite. Therefore, an instanton solution with one flat brane exists.

The evolution of a brane universe after creation is also given by analytic continuation of the Euclidean metric

$$ds_E^2 = dr^2 + b_0^2 e^{2r/l_\pm}\left(d\tau^2 + dx^2 + dy^2 + dz^2\right) \quad (31)$$

by the Wick rotation. Substituting $\tau \to it$, we obtain

$$ds^2 = dr^2 + b_0^2 e^{2r/l_\pm}[-dt^2 + dx^2 + dy^2 + dz^2]\,. \quad (32)$$

We recover the 4D Minkowski spacetime.

## 3.3. AdS brane instanton ($k = -1$)

In the case of an 'open' brane model ($k = -1$), the solution of Eq. (15) is given as

$$b(r) = l_\pm \cosh\left(\frac{r}{l_\pm}\right)\,. \quad (33)$$

We impose the junction condition at the boundary ($r = r_1$ and $r_2$). For the same reason as for a flat brane model, this solution does not provide a single brane model.

The tension of the $i$-th brane is determined by the junction condition (13) as

$$\frac{b'}{b}\left(3 - 4\alpha\frac{b'^2}{b^2} - 12\alpha\frac{1}{b^2}\right) = \mp\frac{\bar{\lambda}_i}{2}. \quad (34)$$

Substituting Eq. (33) into Eq. (34), we obtain

$$\begin{aligned} \bar{\lambda}_i^{(\pm)} &= (-1)^i \frac{2}{l_\pm}\left[(2 \pm \eta)\frac{\sinh(r_i/l_\pm)}{\cosh(r_i/l_\pm)}\right. \\ &\quad \left. - 2(1 \mp \eta)\frac{\sinh(r_i/l_\pm)}{\cosh^3(r_i/l_\pm)}\right]. \end{aligned} \quad (35)$$

In the limit of $r \to \infty$, we recover the critical tension (23), since the curvature of the brane vanishes. Furthermore, if we divide the tension (35) into two parts, $\bar{\lambda} = \bar{\lambda}_{\rm cr} + \triangle\bar{\lambda}$, we find that $\triangle\bar{\lambda}$ is always negative for $0 \le \eta < 1$ Since this tension gives a negative effective cosmological constant on the brane, the brane is anti-de Sitter spacetime.

The total action ($S_E = S_E^{\rm bulk} + S_E^{\rm brane}$) is given by

$$\begin{aligned} S_E &= -\frac{3V_4^{(-)}}{\kappa_5^2} l_\pm^3 \left\{\left[(-2 \pm \eta)\cosh\left(\frac{r_2}{l_\pm}\right)\sinh\left(\frac{r_2}{l_\pm}\right)\right.\right. \\ &\quad \left.\left. + (2 \mp 3\eta)\left(\frac{r_2}{l_\pm}\right)\right] - [r_2 \to r_1]\right\}. \end{aligned} \quad (36)$$

Here $V_4^{(-)}$ is the volume of a 4D manifold with $k = -1$. Again we do not have any minimum in this action.

We cannot give any prediction for a created brane spacetime.

To discuss the evolution of the brane universe after creation, we have to perform the analytically continuation of the Euclidian space:

$$ds_E^2 = dr^2 + l_\pm^2 \cosh^2(r/l_\pm) ds_{E,4}^2 \tag{37}$$

with

$$ds_{E,4}^2 = d\chi^2 + \sinh^2\chi \left( d\psi^2 + \sin^2\psi d\Omega_{(2)}^2 \right) . \tag{38}$$

By the double Wick rotations [? ], i.e., $\chi \to i(t+\pi/2)$ and $\psi \to i\phi$, we obtain the metric of the brane as

$$ds_4^2 = -dt^2 + \cos^2 t \left( d\phi^2 + \sinh^2\phi d\Omega_{(2)}^2 \right) , \tag{39}$$

which gives the AdS spacetime.

## 4. Trace Anomaly

In the previous section, although we can find instanton solutions for several situations, we cannot provide the most plausible state of the brane universe because the action does not give any minimum value. If we choose the critical tension to obtain a zero cosmological constant, only a flat brane instanton is possible. The size of the brane universe, however, is not determined. Any distance between two branes is possible. We again lose the predictability. In this section we discuss another effect, which may predict the initial state of the brane universe. In a curved space time, we know that even in the absence of classical gravitational action, quantum fluctuation of matter fields provides a nontrivial gravitational action through a trace anomaly term $\langle \tau_{\mu\nu} \rangle$. In the case of free, massless, and conformally invariant fields, these quantum corrections take a simple form [13]. These terms were discussed in the context of an inflationary scenario [14] and of the creation of the universe [7, 15]. In [7], assuming the critical tension, the size of the brane universe is fixed. In the present case, we also take into account the contribution of the trace anomaly, our junction condition is modified as

$$\begin{aligned} [B_{\mu\nu}]^\pm &\equiv \left[ -\frac{b'}{b} \left\{ 3 - 4\alpha \left( \frac{b'^2}{b^2} - 3\frac{k}{b^2} \right) \right\} \right]^\pm h_{\mu\nu} \\ &= -\kappa_5^2 \left( \tau_{\mu\nu} + \langle \tau_{\mu\nu} \rangle \right) , \end{aligned} \tag{40}$$

where $\tau_{\mu\nu}$ is the energy-momentum tensor of the brane matter fields and $\langle \tau_{\mu\nu} \rangle$ is its trace anomaly term including the tension of the brane, which is given by

$$\langle \tau_{\mu\nu} \rangle = -\lambda h_{\mu\nu} + H_{\mu\nu}^{(1)} + H_{\mu\nu}^{(3)}. \tag{41}$$

$H_{\mu\nu}^{(1)}$ and $H_{\mu\nu}^{(3)}$ take the following forms:

$$\begin{aligned} H_{\mu\nu}^{(1)} &= -k_1 \Big( 2RR_{\mu\nu} - \frac{1}{2} h_{\mu\nu} R^2 - 2\nabla_\mu \nabla_\nu R \\ &\quad + 2h_{\mu\nu} \nabla^\alpha \nabla_\alpha R \Big) , \end{aligned} \tag{42}$$

$$\begin{aligned} H_{\mu\nu}^{(3)} &= k_3 \Big( -R_\mu{}^\sigma R_{\nu\sigma} + \frac{2}{3} R R_{\mu\nu} + \frac{1}{2} h_{\mu\nu} R^{\sigma\tau} R_{\sigma\tau} \\ &\quad - \frac{1}{4} h_{\mu\nu} R^2 \Big) , \end{aligned} \tag{43}$$

where $R$ and $R_{\mu\nu}$ are the 4D scalar curvature and Ricci tensor, respectively. The coefficient $k_1$ may not appear in $\mathcal{N} = 4$ super conformal Yang-Mills theory but can be included to obtain a successful inflationary scenario. $k_3$ is uniquely determined:

$$k_3 = \frac{1}{2880\pi^2} (2N_0 + 11N_{1/2} + 62N_1), \tag{44}$$

where $N_0$, $N_{1/2}$, and $N_1$ are the number of quantum fields with spins 0, 1/2, and 1, respectively.

We shall include the trace anomaly terms for our instanton solutions. By using the metric (8),

we obtain

$$\langle \tau_{\mu\nu} \rangle = -\lambda h_{\mu\nu} - \frac{3k_3}{b_i^4} h_{\mu\nu}. \tag{45}$$

For a flat brane, the trace anomaly terms $H_{\mu\nu}^{(1)}$ and $H_{\mu\nu}^{(3)}$ vanish. Hence the junction condition is

$$\left[ \frac{b'}{b} \left\{ 3 - 4\alpha \left( \frac{b'^2}{b^2} - \frac{3k}{b^2} \right) \right\} \right]^\pm = -\left( \bar{\lambda} + \frac{3\bar{k}_3}{b_i^4} \right) , \tag{46}$$

where $\bar{\lambda} = \kappa_5^2 \lambda$ and $\bar{k}_3 = \kappa_5^2 k_3$.

For a positive tension brane (either a single brane or the second brane of two-brane model), we obtain

$$\bar{\lambda} = 2\frac{b_i'}{b_i} \left\{ 3 - 4\alpha \left( \frac{b_i'^2}{b_i^2} - \frac{3k}{b_i^2} \right) \right\} - \frac{3\bar{k}_3}{b_i^4} . \tag{47}$$

Since a trace anomaly (45) is always negative (or zero for a flat brane), the tension is always below the value obtained previously. It turns out that only a de Sitter brane with $k = 1$ is possible if we adopt the critical tension. We show the tension in terms of including trace anomaly terms in Fig. 2.

$$\begin{aligned} \bar{\lambda}^{(\pm)} &= \frac{2}{l_\pm} \left\{ (2 \pm \eta) + \frac{2(1 \mp \eta)}{\sinh^2(r_i/l_\pm)} \right\} \\ &\quad - \frac{3\bar{k}_3}{l_\pm^4 \sinh^4(r_i/l_\pm)} . \end{aligned} \tag{48}$$

If the tension is the critical value (23), we find a unique solution with a finite radius.

For a two-brane model, if the tension on one brane is critical, the other is not the case. For example, if the tension of the positive tension brane ($r_2$) is critical, an effective cosmological constant on the negative tension brane ($r_1$) is positive. Conversely, if the tension of the negative tension brane ($r_1$) is critical, we find AdS universe on the positive-tension brane. For a single brane model, the radius of the created brane universe is fixed.

Note that since the trace anomaly always vanishes for a flat brane, the flat two-brane model at any distance is possible.

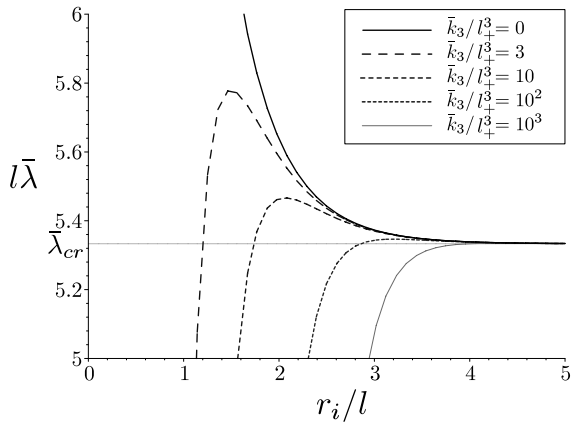


Figure 2: The tension for a de Sitter brane instanton with $\eta = 2/3$ and $\bar{k}_3/l_\pm^3 = 3, 10, 10^2, 10^3$, and 0, where $\bar{k}_3 = 0$ gives the case without the trace anomaly term. This figure corresponds to the cases of plus branche. For minus branch the shape is similar to plus branche's. Since $\lim_{r_i\to\infty}[\bar{\lambda} - \bar{\lambda}_{\rm cr}] = 0+$, we find a unique solution at a finite radius if the tension is the critical value $\bar{\lambda}_{\rm cr}$.

## 5. CONCLUSION

We have presented instanton solutions in the model with a Gauss-Bonnet term. If a brane is a closed universe, we find both two-brane and single-brane de Sitter brane instantons. For a flat brane universe, we can also construct a two-brane instanton solution. As for a single-brane model, a compact bulk spacetime with a single brane is not possible because a flatness of the brane is not consistent with a no-boundary boundary condition. However, the RS II type instanton with a non-compact bulk spacetime is allowed because its Euclidean action is finite. For an anti-de Sitter brane instanton with negative curvature, we can also construct only a two-brane instanton solution. In this case, the RS II type instanton does not exist because the Euclidean action diverges.

Although we find instanton solutions in the model with a Gauss-Bonnet term, we cannot predict the initial state of the brane universe. This is because the Euclidian action has no minimum value. For a flat and an anti-de Sitter brane, the volume of a created brane universe $V_4^{(k)}$ is not fixed as well. $V_4^{(k)}$ can be arbitrary.

As for the tension of a brane, although the critical tension requires fine-tuning, we need such a choice to explain the present small value of the universe. Such a tuning could be obtained in some super symmetric theories, such as the Horava-Witten model. Here we assume such a tuned value, i.e., the critical tension. Then we also include trace anomaly terms on the brane. In this case, we can predict the size of the universe. If we have a single-brane universe with a no-boundary boundary condition, the created universe is in de Sitter phase and naturally evolves into the inflationary stage.

Although we construct brane instanton solutions, in order to predict the initial state of created brane universe, we need to include other important effects, such as the Casimir energy, which are not taken into account here. These issues are left for future study.

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