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CP^{N-1} Harmonic Maps and the Weierstrass System

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We present a Weierstrass-like system of equations corresponding to the CP^{N-1} harmonic maps. This system consists of four first-order equations for three complex functions which are shown to be equivalent to the CP^{N-1} harmonic maps. When the harmonic maps are holomorphic (or antiholomorphic) one of the functions vanishes and the system of equations reduces to the generalisation of the Weierstrass problem given in [1]. We also discuss a geometrical interpretation of our results and show that the induced metric is proportional to the total energy density of the map.

1 Introduction

This talk is based on our recent paper [2] which is shortly to be published in the Journal of Mathematical Physics. In this paper we discuss two interesting ideas: the CP^1 harmonic maps and their relation to the Weierstrass System. We then show how to generalise this idea to the CP^{N-1} case.

The original idea of looking at this problem was initiated a few years ago by Konopelchenko who, together with his collaborators [3,4], introduced the subject of Weierstrass representations of surfaces immersed in multidimensional spaces. This has generated a lot of interest [5,6] and has led to the connection with the CP^{N-1} harmonic maps [1,2].

These generalisations have also lead to the study of immersed surfaces, whose metric is related to the properties of the corresponding harmonic maps. As is well known (see e.g. [7]), in the CP^1 case all harmonic maps (from S^2) are holomorphic (or antiholomorphic). In the CP^1 case the induced metric is characterised by the holomorphic component of the energy thus showing that, in this case, this characterisation is complete.

Our generalisation [2] covers the case of both holomorphic and non-holomorphic maps for, as is known [7], in the CP^{N-1} case (for $N > 2$) there are harmonic maps which are not holomorphic (or antiholomorphic). We explain in some detail, how when the maps are holomorphic, our generalisation reduces to the one given before [1].

In the next section we briefly review the CP^{N-1} harmonic maps (using the formalism as given in [7]) and in the following sections relate these maps to our version of the Weierstrass problem.

2 CP^{N-1} harmonic maps

CP^{N-1} harmonic maps are maps

$$S^2 \to CP^{N-1} \tag{1}$$

given by the stationary points of the energy

$$\mathcal{L} = \int L \, dx \, dy, \tag{2}$$

where

$$L = \frac{1}{4}(D_\mu z)^\dagger \cdot D_\mu z. \tag{3}$$

The differential operator D_μ in (3) acts on $\psi : S^2 \to CP^{N-1}$ as follows:

$$D_\mu \psi = \partial_\mu \psi - \psi(z^\dagger \cdot \partial_\mu z). \tag{4}$$

Here, $\mu = 1, 2$, and denotes the space coordinates x and y. Moreover, z is a vector field with N components, $z = (z^1, \dots, z^N)$, which is normalised to 1; i.e. we have

$$z^\dagger \cdot z = 1. \tag{5}$$

Note that for $N = 2$, i.e. for $CP^1 \sim S^2$, we can introduce a complex field W

$$z = \frac{(1, W)}{\sqrt{1 + |W|^2}}. \tag{6}$$

Then, (for CP^1) the Euler Lagrange equations describing harmonic maps are given by

$$\partial \bar{\partial} W - 2\bar{W} \frac{\partial W \bar{\partial} W}{|W|^2 + 1} = 0. \tag{7}$$

where $W = W(\zeta, \bar{\zeta})$ and where

$$\partial = \frac{\partial}{\partial(x + iy)} = \frac{\partial}{\partial \zeta}, \qquad \bar{\partial} = \frac{\partial}{\partial \bar{\zeta}}. \tag{8}$$

Clearly, solutions of (7) are given by

$$W = W(\zeta) \tag{9}$$

or

$$W = W(\bar{\zeta}) \tag{10}$$

i.e. holomorphic and antiholomorphic functions. What is less clear, however, is that all solutions on S^2 are like this; i.e. there are no harmonic functions which are not holomorphic or antiholomorphic.

This, however, is not the case for the CP^{N-1} model [7] for $N > 2$.

3 The Weierstrass system (WS)

To define a WS we consider solutions of a system of equations for 2 complex functions

$$\psi = \psi(\zeta, \bar{\zeta}) \tag{11}$$

and

$$\phi = \phi(\zeta, \bar{\zeta}), \tag{12}$$

which satisfy the following set of first-order equations:

$$\partial\psi = p\phi, \qquad \bar{\partial}\phi = -p\psi, \qquad p = |\phi|^2 + |\psi|^2. \tag{13}$$

Note that $\bar{\partial}\psi$, or $\partial\phi$ have not been specified.

A natural question then arises. Are these two problems related? Obviously the answer is YES. To see this we put

$$W = \frac{\psi}{\bar{\phi}} \tag{14}$$

and then find that

$$\psi = W\frac{(\bar{\partial}\bar{W})^{\frac{1}{2}}}{1+|W|^2}, \qquad \phi = \frac{(\partial W)^{\frac{1}{2}}}{1+|W|^2} \tag{15}$$

satisfy (13).

Moreover, we can also show that the equations of the two systems are fully equivalent.

4 Geometry

To discuss geometrical aspects of the Weierstrass problem [3], and to relate it to the properties of the CP^1 maps, we introduce 3 real functions

$$X_1 = i\int_\gamma [\bar{\psi}^2 + \phi^2]d\zeta - [\psi^2 + \bar{\phi}^2]d\bar{\zeta}, \tag{16}$$

$$X_2 = \int_\gamma [\bar{\psi}^2 - \phi^2]d\zeta + [\psi^2 - \bar{\phi}^2]d\bar{\zeta}, \tag{17}$$

and

$$X_3 = -2\int_\gamma \bar{\psi}\phi d\zeta + \psi\bar{\phi}d\bar{\zeta}, \tag{18}$$

where γ is any curve from a fixed point to ζ.

It is now easy to show that if ψ and ϕ satisfy the equations of the WS then X_i do not depend on the details of the curve γ but only on its endpoints.

Next we treat X_i as components of $\vec{r} = (X_1, X_2, X_3)$ and introduce the metric

$$g_{\zeta\zeta} = (\partial\vec{r}, \partial\vec{r}), \qquad g_{\bar{\zeta}\bar{\zeta}} = (\bar{\partial}\vec{r}, \bar{\partial}\vec{r}), \qquad g_{\zeta\bar{\zeta}} = (\partial\vec{r}, \bar{\partial}\vec{r}). \tag{19}$$

Then, the detailed calculations show that only $g_{\zeta\bar{\zeta}}$ is non-zero and that

$$g_{\zeta\bar{\zeta}} = \frac{|\partial W|^2}{(1+|W|^2)^2} = |Dz|^2. \tag{20}$$

Here D denotes D_μ evaluated with respect to ζ.

Note that, as all harmonic CP^1 maps on S^2 satisfy $W = W(\zeta)$ and $g_{\zeta\bar{\zeta}}$ is the total energy of the map (for antiholomorphic maps we take complex conjugates).

5 The CP^{N-1} case. Some general points

Next we consider the general CP^{N-1} case. Now we have more components, ie if we define $z = \frac{f}{|f|}$ then f has N components.

Note also that harmonic maps are **not** necessarily holomorphic or antiholomorphic. ie the equation

$$\left(1 - \frac{f \otimes f^\dagger}{|f|^2}\right) \left[\partial\bar{\partial} f - \partial f \frac{(f^\dagger \cdot \bar{\partial} f)}{|f|^2} - \bar{\partial} f \frac{(f^\dagger \cdot \partial f)}{|f|^2}\right] = 0, \tag{21}$$

has solutions other than $f = f(\zeta)$ or $f = f(\bar{\zeta})$.

So what should we do? What should we take for functions ψ and ϕ and how many of them should we use? And how to avoid, in the geometrical interpretation, getting only the holomorphic component of the map which was sufficient in the CP^1 case but which is not for CP^{N-1}.

6 Our procedure [2]

To discuss our procedure [2] it is convenient to rewrite the equations of the CP^{N-1} model in terms of the 'projector formalism' (see e.g. [7]). Thus we introduce

$$P = \frac{f \otimes f^\dagger}{|f|^2}, \qquad P^\dagger = P, \qquad P^2 = P. \tag{22}$$

Then, as is easy to check, the equations for the harmonic maps, i.e. (21) are equivalent to

$$[\partial\bar{\partial} P, P] = 0. \tag{23}$$

However, we can rewrite (23) as a conservation law:

$$\partial[\bar{\partial} P, P] + \bar{\partial}[\partial P, P] = 0, \tag{24}$$

i.e.

$$\partial K - \bar{\partial} K^\dagger = 0, \tag{25}$$

where

$$K = [\bar{\partial} P, P] = \frac{\bar{\partial} f \otimes f^\dagger - f \otimes \bar{\partial} f^\dagger}{|f|^2} + \frac{f \otimes f^\dagger}{|f|^4}\left[(\bar{\partial} f^\dagger \cdot f) - (f^\dagger \cdot \bar{\partial} f)\right]. \tag{26}$$

To proceed further it is convenient to put

$$K = M + L, \tag{27}$$

where

$$M = (1 - P)\frac{\bar{\partial} f \otimes f^\dagger}{|f|^2} = (1 - P)\bar{\partial} P \tag{28}$$

and

$$L = -\frac{f \otimes \bar{\partial} f^\dagger}{|f|^2}(1 - P) = -\bar{\partial} P(1 - P). \tag{29}$$

Next we observe that our matrices M and L satisfy the conservation laws separately, i.e. we have

$$\partial M = \bar{\partial} M^{\dagger} \tag{30}$$

and

$$\partial L = \bar{\partial} L^{\dagger}. \tag{31}$$

Note that the two conservation laws are not really independent; they differ from each other by a total divergence (as was drawn to our attention by A. Mikhailov [9]). To see this note that as P is a projector we have

$$(1-P)P = P(1-P) = 0. \tag{32}$$

Then

$$0 = \bar{\partial}\{(1-P)P\} = -\bar{\partial}PP + (1-P)\bar{\partial}P = -\bar{\partial}PP + M \tag{33}$$

and

$$0 = \bar{\partial}\{P(1-P)\} = \bar{\partial}P(1-P) - P\bar{\partial}P = -L - P\bar{\partial}P. \tag{34}$$

Hence

$$M = \bar{\partial}PP, \qquad L = -P\bar{\partial}P \tag{35}$$

and so we see that

$$M - L = \bar{\partial}PP + P\bar{\partial}P = \bar{\partial}P^2 = \bar{\partial}P \tag{36}$$

thus showing that

$$M = L + \bar{\partial}P. \tag{37}$$

Substituting this into (30) we obtain (31).

7 Exploitation of the conservation laws

Let us write out the matrix elements of M and L (using the summation convention). They are

$$M_{ij} = \bar{\Phi}_i^2 \bar{f}_j \tag{38}$$

and

$$L_{ij} = -f_i \bar{\varphi}_j^2, \tag{39}$$

where

$$\varphi_i^2 = \frac{1}{A^2}\bar{f}_k F_{ki}, \qquad A = \bar{f}_l f_l \tag{40}$$

and

$$\Phi_i^2 = \frac{1}{A^2} f_k \overline{G_{ki}}. \tag{41}$$

In these expression we have introduced:

$$F_{ij} = f_i \partial f_j - f_j \partial f_i, \tag{42}$$

and

$$G_{ij} = f_i \bar{\partial} f_j - f_j \bar{\partial} f_i. \tag{43}$$

Note that due to the antisymmetry of F_{ij} and of G_{ij} we have constraints; namely,

$$\bar{f}_k \varphi_k^2 = 0, \qquad f_k \Phi_k^2 = 0. \tag{44}$$

Hence only $(N-1)$ functions φ_i^2 and $(N-1)$ functions Φ_i^2 are linearly independent.

Thus we take as our independent functions, say, $\varphi_2^2, \ldots, \varphi_N^2$ and $\Phi_2^2, \ldots, \Phi_N^2$. In addition, we can set, without any loss of generality, $f_1 = 1$.

Next we note that we can invert the expressions for φ_i^2 and find that

$$\partial f_i = A \left[\varphi_i^2 + f_i \bar{f}_k \varphi_k^2 \right] \tag{45}$$

and

$$\bar{\partial} f_i = A \left[\bar{\Phi}_i^2 + f_i\, \bar{f}_k \bar{\Phi}_k^2 \right]. \tag{46}$$

These two equations will form part of our Weierstrass system as we discuss in the next section.

8 Relation to the Weierstrass system

Next we use the equations of the CP^{N-1} model to calculate $\bar{\partial}\varphi_i^2$ and $\bar{\partial}\Phi_i^2$. We find that

$$\bar{\partial}\varphi_i^2 = -A\varphi_i^2(\varphi^{\dagger 2} \cdot f) - f_i \left[\Big(\varphi^{\dagger 2} \cdot \varphi^2) + (f^\dagger \cdot \varphi^2)(\varphi^{2\dagger} \cdot f \Big) \right], \tag{47}$$

and

$$\bar{\partial}\Phi_i^2 = -A\Phi_i^2(f^\dagger \cdot \bar{\Phi}^2) - \bar{f}_i \left[\Big(\Phi^{\dagger 2} \cdot \Phi^2) + (f^\dagger \cdot \bar{\Phi}^2)(\bar{\Phi}^{\dagger 2} \cdot f \Big) \right], \tag{48}$$

which, together with (45) and (46) i.e.

$$\begin{aligned} \partial f_i &= A \left[\varphi_i^2 + f_i \bar{f}_k \varphi_k^2 \right], \\ \bar{\partial} f_i &= A \left[\bar{\Phi}_i^2 + f_i \bar{f}_k \bar{\Phi}_k^2 \right], \end{aligned}$$

constitute our **generalisation** of the Weierstrass Problem [2]. Here, as usual, $A = 1 + (f^\dagger \cdot f)$ and all indices, and summations, now go only over $(2, \ldots, N)$.

Note that our procedure has given us four sets of equations for three sets of complex functions, f_i, φ_j and Φ_k. The equations fall into two subclasses (those involving ∂f_i and φ_j and those involving $\partial \bar{f}_i$ and Φ_j). Both sets are equivalent to the same equations for f_i. Note also that instead of taking f_i we could have introduced new functions ψ_i and Ψ_i by, say, $\psi_i = f_i \bar{\varphi}_i$ and $\Psi_i = f_i \bar{\Phi}_i$. Then our sets of functions would have effectively decoupled.

Let us observe that when f_i are holomorphic; i.e. $\bar{\partial} f_i = 0$ then $f^\dagger \cdot \bar{\Phi}^2 = 0$, implying that $|\Phi^2|^2 = 0$. Thus $\Phi_i^2 = 0$ (in analogy with the original Weierstrass system where we had only ϕ and $\psi = f\bar{\phi}$).

9 Geometrical aspects

Now we follow the procedure from Section 4 and introduce real functions X_i which will be treated as components of a vector in a larger dimensional space. To do this we introduce the matrices

$$V = \int_\gamma M d\bar{\zeta} + \int_\gamma M^\dagger d\zeta \tag{49}$$

and

$$W = \int_\gamma L d\bar{\zeta} + \int_\gamma L^\dagger d\zeta \tag{50}$$

and then, for X_i, we take individual entries of each matrix. As $\mathrm{Tr}\, M = \mathrm{Tr}\, L = 0$ V and W have, each, only $N^2 - 1$ independent entries so our construction gives us two vectors in R^{N^2-1}.

Note that W and V do not depend on the contours of integration γ. This, as before, follows from the conservation laws.

Should we consider the entries of W and V separately and so treat our pair of vectors as one vector in a higher dimensional space? This would be a possibility but we do not have to go that far. The reason for this is that, as can be seen by direct calculations, for the solutions (CP^{N-1} harmonic maps) can add our two vectors and obtain a vector which leads to an induced metric with good properties.

Thus we consider

$$X = \int_\gamma (M + L) d\bar{\zeta} + \int_\gamma (M^\dagger + L^\dagger) d\zeta. \tag{51}$$

Then we define the induced metric

$$g_{\alpha\beta} = \sum_{lk} \frac{\partial X_{kl}}{\partial \alpha} \frac{\partial X_{lk}}{\partial \beta}, \tag{52}$$

where α and β stand for ζ or $\bar{\zeta}$.

We find that that for this metric we have

$$g_{\zeta\bar{\zeta}} = |Dz|^2 + |\bar{D}z|^2, \tag{53}$$

which is, of course, the total energy of the harmonic map, while the other components, $g_{\zeta\zeta}$ and $g_{\bar{\zeta}\bar{\zeta}}$, vanish.

The demonstration of this fact is rather complicated as it involves the details of some of the very specific properties of the CP^{N-1} harmonic maps. For more details see [2].

10 Conclusions and some general comments

In this talk we have presented a possible generalization of the Weierstrass problem [2]. Let us stress that this generalization involves more functions than those appearing in the original Weierstrass problem which is very special. However, the original problem has depended on the fact that all harmonic maps from S^2 to CP^1 are either holomorphic or antiholomorphic – which is not the case for CP^{N-1}, when $N > 2$. And even for CP^1 one had to adopt different approaches to holomorphic and antiholomorphic maps (namely, for antiholomorphic maps we had to complex conjugate them – to turn them into holomorphic ones). This is not the case in our construction which holds for all maps.

We can generalise our procedure to other manifolds, such as other Grassmanian models. This involves changing the rank of the projector P in (22). A more interesting generalisation involves establishing the relation to other approaches – say, a lá Bobenko and his collaborators (see e.g. [10]). Such an approach is currently under active study and we hope to report on it in the near future [11].

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