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Spinor Representation of Lie Algebra for Complete Linear Group

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Spinor representation of group $GL(4,\mathbb{R})$ on special spinor space is developed. Representation space has a structure of the fiber space with the space of diagonal matrices as the base and standard spinor space as typical fiber. Non-isometric motions of the space-time entail spinor transformations which are represented by translation over fibering base in addition to standard $Spin(4,\mathbb{C})$ representation.

1 Introduction

Spinor representation of the group $GL(4,\mathbb{R})$ is needed for correct description of the Fermi fields on Riemann space, such as the space-time of general relativity. It is used for two purposes: to define the connectivity and covariant derivative of spinor field and to define the Lie derivative. Recent publications [1–3] have reminded of this problem. The important problem for definition of Fermi fields on Riemann space is that transformation properties of Dirac equation correspond [4] to $Spin(3,1)$ representation of Lorentz group $SO(3,1)$ only, not the full linear group $GL(4,\mathbb{R})$.

Covariant derivative definition and field equations based on it can be defined by the field of orthonormal basis – tetrad description of curve geometry. In such way the tetrad connectivity is a member of Lorentz group and generates the spinor connectivity as standard $Spin(3,1)$ representation. Spinor representation of group $GL(4,\mathbb{R})$ is needed for investigation of the spinor field symmetry as realization of the space-time symmetry.

In the case when the space-time symmetry subgroup G is different from $SO(3,1)$, the subgroup spinor representation of that symmetry cannot be realized as $Spin(3,1)$ subgroup and, one needs the spinor representation of G. As example we can take the standard model of Universe and its $G(6)$ group of symmetry. It contains the subgroup $G(3)$ of isotropy – subgroup of Lorentz group $SO(3,1)$, and subgroup $G(3)$ of translations. The latter is not a part of the Lorentz group and we can describe translation properties of spinor field (i.e. electron) through spinor representation of group $GL(4,\mathbb{R})$ only.

Here we give results of investigations in special construction for the spinor field on the space-time of general relativity. This is an extension of our investigation [5] of spinor representation for full linear group $GL(4,\mathbb{R})$.

2 Standard construction on Riemann space

For each point of the space-time one constructs the orthonormal basis $\overset{k}{e}_{\mu}(x)$ such that scalar products are

$$\overset{k}{e}_{\mu}(x)\overset{m}{e}_{\nu}(x)g^{\mu\nu} = \eta^{km} = \mathrm{diag}\,(1,-1,-1,-1). \tag{1}$$

Each basis vector is represented by Dirac matrix:

$$\overset{k}{e} \Rightarrow \gamma^k, \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}. \tag{2}$$

Coordinate transformations deal with the coordinate index only:

$$\overset{k}{e}_{\mu'}(x') = \frac{\partial x^\mu}{\partial x^{\mu'}} \overset{k}{e}_\mu(x)$$

and group of invariance for the basis is $SO(3,1)$. This group can be represented by transformations of the spinor field:

$$\overset{k}{e}{}'_\mu(x) = T^k_m \overset{m}{e}_\mu(x) \;\Rightarrow\; \psi'(x) = U(T)\psi(x), \tag{3}$$

$$U = \exp\left(\frac{i}{4} T_{mn}\sigma^{mn}\right), \qquad \sigma^{nm} = \frac{i}{2}(\gamma^n\gamma^m - \gamma^m\gamma^n). \tag{4}$$

The main problem in spinor representation of basis transformations is in the existence of special type of conjugacy for Dirac spinor: spinor $\overline{\psi}$ is conjugated to $\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$ if its components are not conjugated only, but are additionally rearranged by the matrix γ^0:

$$\overline{\psi} = (\,\varphi^* \; \chi^*\,)\,\gamma^0. \tag{5}$$

Each transformation modifying that matrix deforms the norm of the spinor space and invariance loses the physical sense.

Covariant derivative can be defined by means of $SO(3,1)$ representation only. The space-time connectivity for basis $\overset{k}{e}_\mu(x+dx)$ is defined by

$$\overset{k}{e}_\mu(x+dx) = \overset{k}{e}_\mu(x) + dx^\nu \omega^k_{m\nu} \overset{m}{e}_\mu(x) \tag{6}$$

This leads to the spinor connectivity

$$\psi(x+dx) = \psi(x) + \frac{i}{4} dx^\nu \omega^k_{m\nu}\eta_{kn}\sigma^{nm}\psi \tag{7}$$

and the covariant derivative

$$\nabla_\mu\psi(x) = \partial_\mu\psi(x) + \frac{i}{4}\omega^k_{m\mu}\eta_{kn}\sigma^{nm}\psi \tag{8}$$

for spinor field.

This is typical approach to involve Fermi fields in general relativity, but it is inappropriate to define the Lie derivative of spinor. Until one considers Lie derivative along Killing vector only, one can keep to previous representation.

If it is needed to involve Lie derivative along non-Killing vector one has to use a corresponding element of group $GL(4,\mathbb{R})$ being outside the Lorentz group. One, may need such a Lie derivative, for example, in the case of investigation of spinor field time dependence for non-static Universe.

3 Point-to-point transformation

We investigate properties of spinor field with respect to motion of the space-time

$$M : x \to y = m(x). \tag{9}$$

Each map $m(x)$ of this motion generates transformation of coordinate basis

$$y = m(x) \ \Rightarrow \ T^{\mu}_{\nu} = \frac{\partial m^{\mu}(x)}{\partial x^{\nu}}. \qquad (10)$$

When the motion belongs to neighborhood of identity, this transformation takes exponential form

$$T = \exp(m \cdot t), \qquad (11)$$

where

$$t^{\mu}_{\nu} = \frac{\partial \zeta^{\mu}(x)}{\partial x^{\nu}} \qquad (12)$$

and vector $\zeta^{\mu}(x)$ determines the direction of motion.

Derivative of basis along motion is the Lie derivative

$$L_{\zeta}\overset{k}{e}_{\mu}(x) = \zeta^{\nu}(x)\partial_{\nu}\overset{k}{e}_{\mu}(x) + \overset{k}{e}_{\nu}(x)\partial_{\mu}\zeta^{\nu}(x). \qquad (13)$$

It generates the transformation of basis

$$\overset{k}{e}(x + \tau\zeta) = \overset{k}{e}(x) + \tau L_{\zeta}\overset{k}{e}(x), \qquad (14)$$

which can be rewritten as

$$\overset{k}{e}(x + \tau\zeta) = \overset{k}{e}(x) + \tau\zeta^{k}_{m}\overset{m}{e}(x). \qquad (15)$$

After integrating we obtain basis transformation as representation of group $GL(4, \mathbb{R})$

$$\overset{k}{e}(m(x)) = \exp\left(m\zeta^{k}_{m}\right)\overset{m}{e}(x). \qquad (16)$$

Only in the case of $\zeta^{\mu}(x)$ being a Killing vector, this representation can be continued to the spinor transformation. In general case this does not work and it is needed to extend the spinor space.

3.1 Space of diagonal matrices

A transformation from neighborhood of identity can be represented as a product of two isometrics V, U and dilatation

$$\Delta = \begin{pmatrix} \delta_0 & 0 & 0 & 0 \\ 0 & \delta_1 & 0 & 0 \\ 0 & 0 & \delta_2 & 0 \\ 0 & 0 & 0 & \delta_3 \end{pmatrix}, \qquad T = V \cdot \Delta \cdot U. \qquad (17)$$

Both isometrics have spinor representation, but the dilatation has no, because it deforms spinor conjugation. One has to extend the spinor space to represent the subgroup of dilatation.

The subgroup of dilatation is a noncompact Abelian group and has true representations as translations in $\mathbb{R}^4$.

Thus it turns out to be interesting to involve into consideration the space of diagonal matrices D_m

$$D_m = \{\mathrm{diag}\,(d_0, d_1, d_2, d_3) : d_0 \cdot d_1 \cdot d_2 \cdot d_3 \neq 0\}\,. \qquad (18)$$

This space realizes representation of the dilatation subgroup Δ which is extended to the representation of group $G(4,\mathbb{R})$ in such a way:

A point d of D_m is transformed by the element of group g to a symmetric matrix d_g which has the diagonal form d'. This diagonal matrix determines the reflex of d trough transformation $T_g(d) = d'$ and determines also unique element Δ_g of dilatation subgroup. Left isometrics V_g is exactly the same as d_g transformation to d' and right isometrics U_g can by restored uniquely trough

$$U_g = \Delta_g^{-1} V_g^{-1} T_g. \tag{19}$$

3.2 Spinor fiber space

Now we construct for each point d from space of diagonal matrices D_m the spinor space $Spin(4,\mathbb{C})$ with anticommutator

$$\begin{aligned} &\gamma^n\gamma^m + \gamma^m\gamma^n = 2d^{mn}, \\ &d^{mn} = \mathrm{diag}\,(d_0, d_1, d_2, d_3), \qquad (\gamma^0)^2 = d_0 \end{aligned} \tag{20}$$

and with conjugation

$$\overline{\psi} = (\,\varphi^*\ \chi^*\,)\gamma^0. \tag{21}$$

Each spinor space $Spin(4,C)$ realizes spinor representation of isometric group $SO(3,1)$ for metrics d^{mn}. All spinor spaces are isomorphic and can be attached to fiber space with base D_m.

Now non-isometric motion $M : x \to y = m(x)$, for each point x from space-time, which has exponential form (11) is represented as product of two isometrics

$$V_g = \exp(v_m \cdot v_g), \qquad U_g = \exp(u_m \cdot u_g) \tag{22}$$

and dilatation

$$\Delta_g = \exp(d_m \cdot \delta_g) \tag{23}$$

as matrix exponent

$$T_g = \exp(v_m \cdot v_g) \cdot \exp(d_m \cdot \delta_g) \cdot \exp(u_m \cdot u_g). \tag{24}$$

The motion T_g is represented on fiber spinor space in three steps:

1. Right isometrics U_g in start point d

$$U_g : \psi(x;d) \;\Rightarrow\; \exp(u_m \cdot u_g(x))\psi(x;d); \tag{25}$$

2. Translation from start point d to end point $d + \delta_g$ over the base of fiber spinor space and to end point $m(x)$ over space-time

$$\Delta_g : \psi(x;d) \;\Rightarrow\; \exp(u_m \cdot u_g(x))\psi(x;d+\delta_g); \tag{26}$$

3. Left isometrics V_g in end point $d + \delta_g$

$$V_g : \exp(u_m \cdot u_g(x))\psi(x;d+\delta_g) \;\Rightarrow\; \exp(v_m \cdot v_g(m(x)))\exp(u_m \cdot u_g(x))\psi(x;d+\delta_g) \tag{27}$$

for the translated spinor.

As result we have the representation

$$T : \psi(x;d) \;\Rightarrow\; \psi_g(m(x);d) = \exp(v_m \cdot v_g(m(x)))\exp(u_m \cdot u_g(x))\psi(x;d+\delta_g), \tag{28}$$

which preserves the spinor norm, if the measure function on the fibering base is translationally invariant.

4 Example: Lie transformation along time direction

As an example we consider the Lie transformation along time for standard model of Universe. Metrics of space-time for this model can be written as

$$ds^2 = dt^2 - R^2(t)dl^2, \tag{29}$$

where dl^2 is the metrics of corresponding space. Transformation from t_1 to t_2 is dilatation with matrix

$$T = \frac{Rt(t_2)}{R(t_1)} \,\mathrm{diag}\,(0, 1, 1, 1) \tag{30}$$

and is represented simply as translation through base of the spinor fiber space. Corresponding Killing vector acts on the spinor field as derivative along direction $(0, 1, 1, 1)$ over base D_m

$$L_t\psi(d; t, x) = \frac{\partial}{\partial t}\psi(d; t, x) + \left(\frac{\partial}{\partial d_1} + \frac{\partial}{\partial d_2} + \frac{\partial}{\partial d_3}\right)\psi(d; t, x). \tag{31}$$

5 Conclusion

- We have developed the special spinor space which represents the full linear group $GL(4, \mathbb{R})$. It has a structure of the fiber space with the space of diagonal matrices as the base and standard spinor space as typical fiber.
- Non-isometric motions of the space-time entail spinor transformations which are represented by translation over fibering base in addition to standard $Spin(4, \mathbb{C})$ representation.
- Until we do not use the non-isometric motion, spinor fields without overlapping on the base of spinor fiber space are independent. Moreover, each such field can be represented as simple spinor space $\psi(x) \Rightarrow \int \psi(x; d)d^4d$.
- Only if one uses the non-isometric motion of space-time, it is essential to consider the fibering of the spinor space.

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