The Role of Symmetry in the Regularity Properties of Optimal Controls

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The role of symmetry is well studied in physics and economics, where many great contributions have been made. With the help of Emmy Noether's celebrated theorems, a unified description of the subject can be given within the mathematical framework of the calculus of variations. It turns out that Noether's principle can be understood as a special application of the Euler–Lagrange differential equations. We claim that this modification of Noether's approach has the advantage to put the role of symmetry on the basis of the calculus of variations, and in a key position to give answers to some fundamental questions. We will illustrate our point with the interplay between the concept of invariance, the theory of optimality, Tonelli existence conditions, and the Lipschitzian regularity of minimizers for the autonomous basic problem of the calculus of variations. We then proceed to the general nonlinear situation, by introducing a concept of symmetry for the problems of optimal control, and extending the results of Emmy Noether to the more general framework of Pontryagin's maximum principle. With such tools, new results regarding Lipschitzian regularity of the minimizing trajectories for optimal control problems with nonlinear dynamics are obtained.

1 Introduction

A natural first step when dealing with an optimal control problem is to apply the existence theorems. For many interesting and important optimal control problems, such as those of the calculus of variations, the existence does not assure the validity of first-order optimality conditions. It may be the case, for example, that the minimizers whose existence is guaranteed by the existence theorem fail to satisfy Pontryagin's maximum principle. Obviously, the study of regularity conditions which close the discrepancy between the hypotheses of existence and necessary optimality theories, is of a crucial importance. This paper contains a survey of the recent results obtained by the author.

We give a new perspective on the results concerning the regularity of the minimizing trajectories. Our claim is that those results are fruit of the existence of certain symmetry properties, and follow from appropriate conservation laws. Such conservation laws are explained by using the recent extensions to optimal control of the classical symmetry theorem of E. Noether.

Emmy Amalie Noether was the one to prove, in 1918, that conservation laws in the calculus of variations are the manifestation of a powerful and universal principle:

> "When a system shows up symmetry with respect to a parameter-transformation, then there exists a conservation law for that system associated with the invariance property".

This assertion comprises all theorems on first integrals known to classical mechanics. Typical application of those results is to lower the order of the differential equations. Here we claim that the conservation laws are also useful for different purposes. We claim that they can be used with success to prove Lipschitzian regularity of the minimizing trajectories.

Noether's principle is so deep and rich, that it can be formalized as a theorem in many different contexts, and in each of such contexts under many different assumptions (see e.g. [12]).

Formulations of Noether's symmetry principle to the more general framework of optimal control have been recently obtained (see e.g. [7, 8, 10]). In this context conservation laws are interpreted as quantities which are preserved along all the Pontryagin extremals. They play here a fundamental role in order to bring together the regularity results in [6, 11].

2 Optimal control and the calculus of variations

The optimal control problem consists in finding a r-vector function $u(\cdot)$, called the control, and the corresponding n-state trajectory $x(\cdot)$, solution of a dynamical system described by n ordinary differential equations under specified boundary conditions, in such a way the pair $(x(\cdot), u(\cdot))$ minimizes a given integral functional:

$$\begin{aligned}
&I[x(\cdot), u(\cdot)] = \int_a^b L\left(t, x(t), u(t)\right) \mathrm{d}t \longrightarrow \min, \\
&\dot{x}\left(t\right) = \varphi\left(t, x\left(t\right), u\left(t\right)\right), \qquad (x(a), x(b)) \in \mathcal{F}, \\
&x\left(\cdot\right) \in \mathcal{X}\left([a,b]\,;\,\mathbb{R}^n\right), \qquad u\left(\cdot\right) \in \mathcal{U}\left([a,b]\,;\,\Omega \subseteq \mathbb{R}^r\right).
\end{aligned} \tag{1}$$

The Lagrangian L and the velocity vector φ are assumed to be smooth functions:

$$C^1 \ni L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}, \qquad C^1 \ni \varphi : [a,b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n.$$

For the particular case when there are no restrictions on the control value, $\Omega = \mathbb{R}^r$, and the control system is just $\varphi(t, x, u) = u$, one obtains the fundamental problem of the calculus of variations ($r = n$),

$$\begin{aligned}
&I[x(\cdot), u(\cdot)] = \int_a^b L\left(t, x(t), u(t)\right) \mathrm{d}t \longrightarrow \min, \\
&\dot{x}(t) = u(t), \qquad (x(a), x(b)) \in \mathcal{F}, \qquad x\left(\cdot\right) \in \mathcal{X}\left([a,b]\,;\,\mathbb{R}^n\right),
\end{aligned}$$

which includes all classical mechanics. Clearly, for the optimal control problem to be well defined, it is crucial to specify the class of functions $\mathcal{X}$ and $\mathcal{U}$. This is addressed in the following section.

3 Logical-deductive approach, discrepancy, and bad behavior

The logical-deductive approach for the resolution of an optimization problem is the following:

1. A solution exists for the problem.
2. The necessary optimality conditions are applicable and they identify certain candidates (in the context of the calculus of variations and optimal control called *extremals*).
3. Elimination, if necessary, identifies the solution or solutions.

The historical facts show that mathematics does not always advance using the logical steps: both the calculus of variations and the mathematical theory of optimal control have born from the study of necessary optimality conditions, the existence question being delayed. In the calculus of variations the first obtained necessary conditions were the Euler–Lagrange differential equations, first proved by L. Euler in 1744, while the existence was only addressed two centuries later, in 1911, by L. Tonelli. The mathematical theory of optimal control, the modern face of the calculus of variations, was born with the Pontryagin maximum principle, proved by V.G. Boltyanski, R.V. Gamkrelidze, and L.S. Pontryagin in 1956 [2], while existence was solved three years later, in 1959, by A.F. Filippov [5]. It is very interesting the explanation why the study of necessary conditions have been first in time. It turns out that existence and necessary optimality theories rely on different classes $\mathcal{X}$ and $\mathcal{U}$:

- In the calculus of variations the existence is assured in the class of absolutely continuous functions ($\mathcal{X} = W_{1,1}$), while the arguments which lead to the necessary conditions demand the minimizing trajectory to be at least Lipschitzian ($\mathcal{X} \subseteq W_{1,\infty}$).
- In optimal control the existence is assured in the class of integrable controls ($\mathcal{U} = L_1$), while the classical formulation of the Pontryagin maximum principle assumes that the optimal controls are essentially bounded ($\mathcal{U} \subseteq L_\infty$).

The discrepancy between the hypotheses of existence and optimality theories is not the result of technical details, but has an intrinsic nature. The gap is a reality, even for problems very innocent in aspect. For example, the minimizers predicted by the existence theory can violate the necessary conditions even when the Lagrangian L is a polynomial and φ is linear. The first problem showing this possibility of bad behavior was proposed by J. Ball and V. Mizel in 1985.

Example 1 (cf. [1]). For the Ball–Mizel problem,

$$\int_0^1 \left(\left| x^3 - t^2 \right|^2 |u|^{14} + \varepsilon \, |u|^2 \right) dt \to \min,$$
$$\dot{x}(t) = u(t), \qquad x(0) = 0, \qquad x(1) = k,$$

it happens:

- All hypotheses of Tonelli's existence theorem are satisfied.
- For some choices of the constants ε and k one has the unique optimal control $u(t) = k\, t^{-1/3}$.
- Pontryagin maximum principle (Euler–Lagrange equation in integral form) is not satisfied since $\dot{\psi}(t) = L_x(t, x(t), \dot{x}(t)) = c\, t^{-4/3}$ is not integrable.

The study of conditions which assure the regularity of the minimizers is important, and excludes occurrence of bad behavior.

4 Regularity of minimizers

A *regularity result* is any assertion assuring that the solution belongs to a class of functions more restrict than the one considered in the formulation of the problem (e.g., $\tilde{x}(\cdot) \in W_{1,\infty}$ instead of $W_{1,1}$; $\tilde{u}(\cdot) \in L_\infty$ instead of L_1). The objective is to find conditions beyond those of the existence theory, assuring that all the minimizers satisfy the standard necessary optimality conditions (e.g. the Pontryagin maximum principle or the Euler–Lagrange differential equations). A profound search in the literature shows that:

- The calculus of variations is very rich in regularity results.
- Regularity results for optimal control problems are a rarity.

In this paper we comment on a new approach to establish regularity properties for the optimal controls. Our approach has its genesis in a very beautiful regularity theorem of 1985, due to F.H. Clarke and R.B. Vinter.

Theorem 1 (cf. [4]). *Under the coercivity and convexity conditions of Tonelli's existence theorem, all the minimizing trajectories $\tilde{x}(\cdot)$ of the* autonomous *fundamental problem of the calculus of variations,*

$$I[x(\cdot), u(\cdot)] = \int_a^b L(x(t), u(t)) \, \mathrm{d}t \longrightarrow \min,$$
$$\dot{x}(t) = u(t), \qquad (x(a), x(b)) \in \mathcal{F}, \qquad x(\cdot) \in W_{1,1}([a, b]; \mathbb{R}^n),$$

are Lipschitzian: $\tilde{x}(\cdot) \in W_{1,\infty} \subset W_{1,1}$.

Theorem 1 is not valid for autonomous optimal control problems with dynamics different from $\dot{x}(t) = u(t)$. We also remark that Theorem 1 is only valid in the autonomous situation, that is, in the case the Lagrangian L and the velocity vector φ do not depend explicitly on the time variable t.

The proof of Clarke and Vinter is very technical, and has the focus on the use of nonsmooth analysis. This, in our opinion, hides the crucial point and the natural question to be asked.

5 Autonomous problems ... why special?

For autonomous problems, the Hamiltonian $H(x, u, \psi_0, \psi) = \psi_0 L(x, u) + \psi \cdot \varphi(x, u)$ is preserved along the Pontryagin extremals $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of the optimal control problem:

$$H(x(t), u(t), \psi_0, \psi(t)) \equiv \text{const}, \qquad t \in [a, b]. \tag{2}$$

This is a consequence of the Pontryagin maximum principle, and holds generally when admissible controls are bounded: $u(\cdot) \in L_\infty$. The property is important in many different contexts:

- In the calculus of variations it corresponds to the 2^{nd} *Erdmann necessary condition.*
- In classical mechanics has the meaning of *conservation of energy.*
- In economy is interpreted has the *income/wealth law.*

The starting point for our approach is the observation that Theorem 1 is a straightforward consequence of the coercivity hypotheses of the existence theory, if one uses the fact that, for the fundamental autonomous problem of the calculus of variations, (2) is valid in the class of absolutely continuous admissible trajectories ($u(\cdot) \in L_1$): see [3, pp. 61–63].

6 What about the non autonomous problems?

Property (2) admits a generalization for the problems which depend explicitly on the time variable t. If $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ is a Pontryagin extremal, then $H(t, x(t), u(t), \psi_0, \psi(t))$ is an absolutely continuous function in t, and satisfies the equality

$$\frac{\mathrm{d}H}{\mathrm{d}t}(t, x(t), u(t), \psi_0, \psi(t)) = \frac{\partial H}{\partial t}(t, x(t), u(t), \psi_0, \psi(t)), \tag{3}$$

where on the left-hand side we have the total derivative with respect to t, and on the right-hand side the partial derivative of the Hamiltonian with respect to t. In the calculus of variations (3) corresponds to the classical necessary condition of DuBois–Reymond. We now give a more general formulation.

Theorem 2 (cf. [9]). *If $F(t, x, u, \psi_0, \psi)$ is a real value function; continuously differentiable with respect to t, x and ψ, for fixed u; $\exists\, G(\cdot) \in L_1([a, b]; \mathbb{R})$ such that*

$$\begin{aligned} &\left\|\nabla_{(t,x,\psi)} F(t, x(t), u(s), \psi_0, \psi(t))\right\| \leq G(t) \qquad (s, t \in [a, b]), \\ &F(t, x(t), u(t), \psi_0, \psi(t)) = \sup_{v \in \Omega} F(t, x(t), v, \psi_0, \psi(t)), \end{aligned}$$

then $t \to F(t, x(t), u(t), \psi_0, \psi(t))$ is an absolute continuous function and satisfies the equality

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial \psi} - \frac{\partial F}{\partial \psi} \cdot \frac{\partial H}{\partial x} \tag{4}$$

along all the extremals.

Choosing $F = H$ in Theorem 2 we obtain equality (3). In the classical context, where no dependence exists on the controls, (4) is the well-known relation $\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \{F, H\}$, where $\{F, H\}$ denotes the Poisson bracket of the functions F and H.

7 Conservation laws in optimal control

Theorem 2 is very useful. From it one obtains a necessary and sufficient condition for a function to be a conservation law.

Definition 1. A quantity $F(t,x,u,\psi_0,\psi)$ which is preserved in t along all the Pontryagin extremals $(x(\cdot),u(\cdot),\psi_0,\psi(\cdot))$ of the optimal control problem (1),

$$F(t,x(t),u(t),\psi_0,\psi(t)) = \text{const}, \tag{5}$$

is called a *first integral*. Equation (5) is the corresponding *conservation law*.

Corollary 1. *Under the conditions of Theorem 2, F is a first integral if, and only if,*

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\cdot\frac{\partial H}{\partial \psi} - \frac{\partial F}{\partial \psi}\cdot\frac{\partial H}{\partial x} = 0.$$

Equality (2) is a trivial consequence of Corollary 1:

Example 2. The Hamiltonian H is a first integral if, and only if, $\frac{\partial H}{\partial t} = 0$. This means that the problem must be autonomous in order for the Hamiltonian to be a first integral.

We have reached the point where it is impossible not to think of the celebrated symmetry theorem of Emmy Amalie Noether (1882–1935). In 1918 E. Noether proved that the conservation laws of the calculus of variations are fruit of the symmetries of the problems. For example, the autonomous fundamental problem of the calculus of variations is invariant under time-translations $t \to t+s$. From this invariance it results, from Noether's symmetry theorem, the conservation law $H = \text{const}$.

8 Symmetry in optimal control

To obtain a formulation of the theorem of Emmy Noether in the more general setting of optimal control, we begin by extending the concept of symmetry that one finds in the calculus of variations. We deal with transformations of the optimal control problem depending on time, state, and control variables; and we consider an invariance notion up to addition of gauge terms which, in general, are non-linear with respect to the parameters, time, state, and control variables.

Definition 2. The optimal control problem (1) is said to be invariant under $h^s(t,x,u) = (h_t^s(t,x,u), h_x^s(t,x,u)) \in C^1$ up to $\Phi^s(t,x,u) \in C^1\left([a,b],\mathbb{R}^n,\Omega;\mathbb{R}\right)$, if $h^0(t,x,u) = (t,x)$ and for all $s = (s_1,\ldots,s_\rho)$, $\|s\| < \varepsilon$, and for all $\beta \in [a,b]$, there exists a $u^s(\cdot) \in L_\infty\left([a,b];\Omega\right)$ such that

$$\begin{aligned}
&\int_{h_t^s(a,x(a),u(a))}^{h_t^s(\beta,x(\beta),u(\beta))} L\left(t^s, h_x^s(t^s,x(t^s),u(t^s)), u^s(t^s)\right)\mathrm{d}t^s \\
&\qquad = \int_a^\beta \left(L(t,x(t),u(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\Phi^s(t,x(t),u(t))\right)\mathrm{d}t, \\
&\frac{\mathrm{d}}{\mathrm{d}t^s} h_x^s(t^s,x(t^s),u(t^s)) = \varphi(t^s, h_x^s(t^s,x(t^s),u(t^s)), u^s(t^s)).
\end{aligned}$$

Equivalently, one says that h^s is a *symmetry* for problem (1).

Theorem 3 (cf. [7]). *If the optimal control problem* (1) *is invariant under the transformations $(h_t^s(t,x,u), h_x^s(t,x,u))$ up to $\Phi^s(t,x,u)$, in the sense of Definition 2, then the following ρ conservation laws hold ($k = 1,\ldots,\rho$):*

$$\begin{aligned}
&\psi_0 \frac{\partial}{\partial s_k}\, \Phi^s(t,x(t),u(t))\big|_{s=0} + \psi(t)\cdot\frac{\partial}{\partial s_k}\, h_x^s(t,x(t),u(t))\big|_{s=0} \\
&\qquad - H\left(t,x(t),u(t),\psi_0,\psi(t)\right)\frac{\partial}{\partial s_k}\, h_t^s(t,x(t),u(t))\big|_{s=0} = \text{const}.
\end{aligned} \tag{6}$$

Example 3. The autonomous optimal control problem is invariant under $h_t^s = t+s$ and $h_x^s = x$. Theorem 3 implies the conservation law (2).

It is important to note that the conservation laws (6) are valid along all the Pontryagin extremals $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of the problem: both for normal ($\psi_0 \neq 0$) and abnormal ones ($\psi_0 = 0$). While abnormal extremals are not a possibility for the fundamental problem of the calculus of variations (one can always put the cost multiplier ψ_0 to be -1), they happen to occur frequently in the general situation. The Martinet flat problem of sub-Riemannian geometry, considered in Example 4, is one such case: it admits abnormal minimizers.

For complex situations, the conditions found in [13] help to obtain the invariance transformations. For the present purposes, it is enough to consider the homogeneous case.

Corollary 2. *If there exist constants α, $\beta_1, \ldots, \beta_n$, $\gamma_1, \ldots, \gamma_r \in \mathbb{R}$, such that for all $\lambda > 0$*

$$L\left(\lambda^\alpha t, \lambda^{\beta_1} x_1, \ldots, \lambda^{\beta_n} x_n, \lambda^{\gamma_1} u_1, \ldots, \lambda^{\gamma_r} u_r\right) = \lambda^{-\alpha} L\left(t, x_1, \ldots, x_n, u_1, \ldots, u_r\right),$$
$$\varphi_i\left(\lambda^\alpha t, \lambda^{\beta_1} x_1, \ldots, \lambda^{\beta_n} x_n, \lambda^{\gamma_1} u_1, \ldots, \lambda^{\gamma_r} u_r\right) = \lambda^{\beta_i - \alpha} \varphi_i\left(t, x_1, \ldots, x_n, u_1, \ldots, u_r\right),$$

the following conservation law holds:

$$\sum_{i=1}^{n} \beta_i \psi_i(t) x_i(t) - \alpha H\left(t, x(t), u(t), \psi_0, \psi(t)\right) t \equiv \text{const.}$$

Proof. Conclusion follows from Theorem 3 by choosing $h_t^s = (s+1)^\alpha t$, $h_{x_i}^s = (s+1)^{\beta_i} x_i$, $i = 1, \ldots, n$, and $u_k^s = (s+1)^{\gamma_k} u_k$, $k = 1, \ldots, r$. ■

Example 4. Let us consider the Martinet flat problem of sub-Riemannian geometry:

$$\int_a^b (u_1(t))^2 + (u_2(t))^2 \, dt \longrightarrow \min,$$
$$\dot{x_1}(t) = u_1(t), \qquad \dot{x_2}(t) = u_2(t), \qquad \dot{x_3}(t) = (x_2(t))^2 u_1(t).$$

With $\alpha = 2$, $\beta_1 = \beta_2 = 1$, $\beta_3 = 3$, $\gamma_1 = \gamma_2 = -1$, we get from Corollary 2 the conservation law $\psi_1 x_1(t) + \psi_2(t) x_2(t) + 3\psi_3 x_3(t) - 2Ht \equiv \text{const.}$

The regularity conditions we are looking for, are obtained with the help of Corollary 2.

9 Symmetry and regularity

The regularity results in [4, 6, 11] are obtained using very different techniques. In this paper we argue that from the point of view of symmetries, all the proofs, which are completely different in detail, can be summarized, in the large, by the same steps. In fact the results follow from the same assumption: that under the typical coercivity conditions of the existence theory, the Lipschitzian regularity conditions for the minimizing trajectories $\tilde{x}(\cdot)$ can be obtained from the applicability conditions of the Pontryagin maximum principle to an equivalent auxiliary problem which possess rich symmetry properties. Our general scheme to prove regularity theorems is summarized in the following steps and ingredients:

1. Reduce the problem to an equivalent form (equivalence in the sense of Elie Cartan, 1908) with a richer set of extremals, and possessing an appropriate symmetry.
2. Establish the relationship between the extremals of the problems (equivalence in the sense of Constantin Carathéodory, 1906).

3. Impose the applicability conditions of the maximum principle to the auxiliary problem, and use the conservation law related to it. From the coercivity condition and the Carathéodory equivalence, conclude that all (normal) minimizing controls of the optimal control problem (1) are essentially bounded, and the respective trajectories are Lipschitzian.

Applying such algorithm with the ideas of R.V. Gamkrelidze and Weierstrass, gives an explanation for the results in [6, 11].

9.1 Regularity from time-homogeneity

Following R.V. Gamkrelidze, we make a reduction of problem (1) to an equivalent autonomous time-optimal problem with the controls taking values in a sphere.

Theorem 4 (cf. [6]). *For the case of control-affine dynamics, $\varphi\,(t,x,u) = f\,(t,x) + g\,(t,x)\cdot u$, the coercivity hypotheses; complete rank r of $g(\cdot,\cdot)$; and the growth condition: there exist constants γ, β, η and μ, with $\gamma > 0$, $\beta < 2$ and $\mu \geq \max\{\beta - 2, -2\}$, such that for all $t \in [a,b]$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$,*

$$\left(|L_t| + |L_{x^i}| + \|L\,\varphi_t - L_t\,\varphi\| + \|L\,\varphi_{x^i} - L_{x^i}\,\varphi\|\,\right)\,\|u\|^{\mu} \leq \gamma\,L^{\beta} + \eta,$$

imply that all the minimizers $\tilde{u}\,(\cdot)$ of the optimal control problem, which are not abnormal extremal controls, are essentially bounded on $[a,b]$.

9.2 Regularity from control-homogeneity

Writing the problem in a parametric form, an idea as old as Weierstrass, one gets *control homogeneity.* Applying the general scheme, the following result comes out.

Theorem 5 (cf. [11]). *Under the coercivity hypothesis of the existence theorem, the growth conditions: there exist constants $c > 0$ and k such that*

$$\left|\frac{\partial L}{\partial t}\right| \leq c\,|L| + k, \qquad \left\|\frac{\partial L}{\partial x}\right\| \leq c\,|L| + k, \qquad \left\|\frac{\partial \varphi}{\partial t}\right\| \leq c\,\|\varphi\| + k, \qquad \left\|\frac{\partial \varphi_i}{\partial x}\right\| \leq c\,|\varphi_i| + k;$$

imply that all the minimizing controls $\tilde{u}(\cdot)$ of the optimal control problem (1)*, which are not abnormal extremal controls, are essentially bounded on* $[a,b]$.

10 Conclusions

Regularity theorems are very important because they imply that all the minimizers are Pontryagin extremals. New conditions of Lipschitzian regularity of the minimizing trajectories, to a broad class of problems of optimal control, are established using appropriate conservation laws. The results are valid for nonlinear control systems. Even for linear dynamics, for example for the problems of the calculus of variations, one can deal with situations for which the previously known regularity conditions fail (cf. [6]). Our approach does not require the Lagrangian L and the velocity vector φ to be convex functions with respect to the control variables.

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