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Yangians of Classical Lie Superalgebras: Basic Constructions, Quantum Double and Universal *R*-Matrix

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Some basic results of the theory of Yangians of Lie superalgebras are described. Yangian of basic Lie superalgebra is described as a result of quantization of Lie superbialgebra of polynomial loops. Also, we consider a quantization of Lie superalgebra of twisted currents. Two systems of generators and defining relations are introduced. Equivalence of this systems of generators and defining relations is proved. The notion of double of Yangian and formula for universal R-matrix for double of Yangian are discussed for Yangian of Lie superalgebra $A(m, n)$ type.

1 Introduction

The main goal of this article is to describe the some basic results of the theory of Yangians of Lie superalgebras of classical type. It is well-known that Lie superalgebras of classical type divided into two classes: basic and strange Lie superalgebras. The Lie superalgebras from the first class have characteristics like those of simple Lie algebras. They have a nondegenerate invariant bilinear form (and nontrivial Casimir operator). This characteristic makes possible to define Yangian of basic Lie superalgebra as Yangian of simple Lie algebra. I remind that the notion of Yangian of a simple Lie algebra g was introduced by V. Drinfeld [1] as a deformation of the universal enveloping algebra $U(g[t])$ of a current algebra $g[t]$ and it was extended to the case of a Lie superalgebras of classical type (see [4, 5]). But strange Lie superalgebras P_n, Q_n (see [12]) do not have natural bisuperalgebra structure. But such structure can be defined onto the twisted current superalgebra (see [7]). But we, as distinct from [7] use diagram automorphism and Drinfeld realization of Yangian.

The main tool of applications of the quantum algebras theory in mathematical physics (and physics) is a quantum R-matrix. I remind the definition of an R-matrix [1–3]. An R-matrix is a function $R(u)$ of a complex parameter u with values in $\mathrm{End}\,(V \otimes V)$, where V is a finite-dimensional vector space (superspace) which satisfies the quantum Yang–Baxter equation (QYBE) (or grading quantum Yang–Baxter equation (gQYBE) in the case of superspace):

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v),$$

(here, $R_{12} = R \otimes id \in \mathrm{End}\,(V \otimes V \otimes V)$, etc.) Drinfeld relates to Yangian $Y(g)$ (g is a simple Lie algebra) an element R in some formal completion of $Y(g) \otimes Y(g)$ which is called universal R-matrix (see [1]). Universal R-matrix is defined by the following condition. R-matrix conjugates comultiplication and opposite comultiplication: $\Delta'(a) = R\Delta(u)R^{-1}$. Quantum R-matrix R_V is image of universal R-matrix R under action of the representation operator $\rho_V \otimes \rho_V : Y(g) \otimes Y(g) \to \mathrm{End}\,(V \otimes V)$. The problem of description (or computation) of universal R-matrix is very important as all formulas of quantum R-matrices are consequences of the formula for universal R-matrix. In many cases it reasonable to work with quantum double $DY(g)$ of the Yangian $Y(g)$ if we keep in mind application in quantum field theory. The definition of double for

Hopf algebras was given by V. Drinfeld, explicit formulas for double of Yangian were obtained by F. Smirnov [10] and S. Khoroshkin, V. Tolstoy [9]. The notion of $DY(G)$ for Lie superalgebra G we discuss below (see also [5]). Universal R-matrix for quantum double is a canonical element $R = \sum a_i \otimes a_i'$, where a_i, a_i' are dual bases in $DY(G)$. Yangians of the Lie algebras have been studied at present time in many papers (see [2]). The theory of Yangians for Lie superalgebras (or super Yangians) is less developed than the one for Lie superalgebras (see [4–7]). In this work we made an attempt to develop the theory of Yangians for Lie superalgebras of classical type and generalize some results from [4, 5]. The structure of this work is following. In Section 1 we remind basic definitions of the Lie superalgebras theory. In Section 2 we define bisuperalgebra structure on superalgebras polynomial loops for basic Lie superalgebras and on superalgebra of twisted currents on Lie superalgebra $A(n,n)$. We introduce the main object $Y(G)$, the Yangian of basic Lie superalgebra $Y(G)$ as a quantization of a $U(G[t])$ and the Yangian of strange Lie superalgebra Q_n as a quantization of twisted current Lie superalgebra with values in $A(n,n)$. We describe $Y(G)$ in terms of generators and defining relations. Starting from a system of zero-order generators for which the relations and the comultiplication law are similar to those of the universal enveloping superalgebra, we introduce first-order generators, for which the comultiplication law is more complicated. The remaining relations follow from the compatibility conditions for the algebra and coalgebra structures. From the finite system of generators and relations thus obtained we derive a new system of generators and relations (Section 3) (in the case of the basic superalgebra of the $A(m,n)$ type). In Section 4 we discuss the structure of double and multiplicative formula of universal R-matrix for double of super Yangian $Y(A(m,n))$. The proofs either not adduced or are given only as sketches.

2 Lie superalgebras and superbialgebras

Let us recall some basic definitions from Lie superalgebra theory [8]. Lie superalgebra $G = G_0 \oplus G_1$ is a Z_2-graded algebra with product $[\cdot,\cdot]$, i.e. if $a \in G_\alpha$, $b \in G_\beta$, $\alpha, \beta \in Z_2 = \{0,1\}$, then $[a,b] \in G_{\alpha+\beta}$. A Lie superalgebra G is superalgebra satisfying the following axioms:

$$[a,b] = -(-1)^{\alpha\beta}[b,a], \qquad [a,[b,c]] = [[a,b],c] + (-1)^{\alpha\beta}[b,[a,c]], \qquad a \in G_\alpha, \qquad b \in G_\beta.$$

Let $A = (a_{ij})$ be a matrix of order r and τ a subset of the set $I = \{1,\dots,r\}$ (as a rule $\tau = \{k\}$, $1 \le k \le r$). Let us assume that Cartan matrix A is a symmetrizable matrix and $B = DA$ is a symmetrical matrix and $D = \operatorname{diag}(d_1,\dots,d_r)$, $d_k \neq 0$, is a diagonal matrix. Let $G = G(A,\tau)$ be the Lie superalgebra with generators x_i^+, x_i^-, h_i, $i \in I$ and the following defining relations:

$$\begin{aligned} &\left[x_i^+, x_i^-\right] = \delta_{ij} h_i, \qquad [h_i, h_j] = 0,\\ &\left[h_i, x_i^\pm\right] = \pm b_{i,j} x_j^\pm, \qquad \left(\mathrm{ad}^{n_{ij}} x_i^\pm\right)\left(x_j^\pm\right) = 0, \quad i \neq j, \end{aligned}$$

where $n_{i,j} = 1$ if $b_{i,i} = b_{i,j} = 0$; $n_{i,j} = 2$ if $b_{i,i} = 0$, $b_{i,j} \neq 0$; $n_{i,j} = 1 - 2b_{i,j}/b_{i,i}$, if $b_{i,i} \neq 0$; $\operatorname{ad} a(b) = [a,b]$. Let $\deg(x)$ be a α if $x \in G_\alpha$, $\alpha \in Z_2$.

Let $\Delta = \Delta_0 \cup \Delta_1$ be a root system of G, Δ_i be a root system of G_i, Δ^+, Δ_i^+ be a sets of positive roots of G, G_i respectively, $\Pi = \{\alpha_1,\dots,\alpha_r\}$ be a simple root system. Recall that a simple root α_i is called white if $i \in I \setminus \tau$ and grey or black if $i \in \tau$ and $a_{ii} = 0$ or $a_{ii} = 2$, respectively. Remind also, that a finite-dimensional Lie superalgebra $G = G_0 \oplus G_1$ is of classical type (or classical), if it is simple, and the representation of G_0 in G_1 is completely reducible. We need the description of the Lie superalgebras $A(m,n)$ and Q_n. For $A(m,n)$ the Cartan matrix $A = (a_{i,j})_{i,j=1}^r$, $r = m+n+1$ is the following: $a_{i,i} = 2$, $i \neq m+1$, $a_{m+1,m+1} = 0$, $a_{i+1,i} = -1$ for all i, $a_{i,i+1} = -1$, $i \neq m+1$, $a_{m+1,m} = 1$, and $a_{i,j} = 0$ elsewhere.

For the definition of strange Lie superalgebra Q_n consider diagram automorphism $\sigma : A(n,n) \to A(n,n)$, $\sigma(f_i) = f_{2n+2-i}$, $(f \in \{h, x^+, x^-\})$. The strange classical Lie superalgebra Q_n is the fixed point subalgebra in $A(n,n)$ with respect to the involutive diagram automorphism σ. The subalgebra Q_n has the property that the Cartan matrix H does not coincide with Cartan subalgebras of the even part A_n but admits also an odd part H_1. More precisely, one has $H = H_0 \oplus H_1$, where H_0 is spanned by the $\{h_i\}$ generators and H_1 by the $\{k_i\}$ generators $(1 \le i \le n)$. However, since the $\{k_i\}$ generators are odd, the root generators $x^{\pm}_{\alpha}$ are not eigenvectors of H_1. It is convenient in this case to give the root decomposition with respect to H_0 instead of H. The root system Δ of Q_n coincides then with the root system of A_n. One has $G = G_0 \oplus G_1 = H_0 \oplus (\oplus_{\alpha \in \Delta} G_\alpha)$ with $\dim G_{\alpha \ne 0} = 2$ and $\dim G_{\alpha = 0} = n$. Moreover, the non-zero roots of Q_n are both even and odd. Note also that in $Q_n \otimes Q_n$ are not nontrivial Q_n-invariants.

Let G be a Lie superalgebra with symmetrizable Cartan matrix and nondegenerate Killing form. A bisuperalgebra G is a vector superspace with superalgebra and cosuperalgebra structures compatible in the following sense: the cocommutator map $\varphi : G \to G \otimes G$ must be a 1-cocycle (we assume that G acts on $G \otimes G$ by means of adjoint representations). Let the cocycle φ be a coboundary of element $r \in \bigwedge^2 g$. Consider the current superalgebra $G[t]$ and introduce the cocommutator φ by formula

$$\varphi : a(u) \to [a(u) \otimes 1 + 1 \otimes a(v), r(u,v)],$$

where $r = \frac{t}{u-v}$ and t is a Casimir operator. Cocommutator φ defines on $G[t]$ the bisuperalgebra structure. Let us note that r satisfies the Yang–Baxter equation (or the triangle equation) (see [1]).

Consider the case of Lie superalgebra Q_n. We are going to define the bisuperalgebra structure on the twisted current superalgebra or on the fixed point subalgebra in $A(n,n)[u]$. Let G be a basic superalgebra $A(n,n)$, $\sigma : G \to G$ earlier defined automorphism of order 2, $\sigma^2 = 1$. Then all eigenvalues of σ are ± 1. The automorphism σ is diagonalized and we have the following decomposition: $G = G^0 \oplus G^1$, where $G^j = \mathrm{Ker}\,(\sigma - \epsilon^j I)$, $\epsilon = e^{\pi i} = -1$, $j \in Z_2$; $G^0 = Q_n$. Let us associate subalgebra $G[u]^\sigma$ of algebra $G[u]$ to automorphism σ:

$$G[u]^\sigma = \Big(\bigoplus_{k \in Z_+} (G^0 \otimes t^{2k})\Big) \bigoplus \Big(\bigoplus_{k \in Z_+} (G^1 \otimes t^{2k+1})\Big).$$

Let us note that

$$G[u]^\sigma = \{a(u) \in G[u] : \sigma(a(u)) = a(-u)\}.$$

Let us define bisuperalgebra structure on the $G[u]^\sigma$. Let us introduce

$$r_\sigma(u,v) = \frac{t}{u-v} + \frac{id \otimes \sigma(t)}{u+v},$$

where $t \in G^{\otimes 2}$ is a Casimir operator. It is easy to check (see [7]) that the function $r_\sigma(u,v)$ is antisymmetric and obeys classical Yang–Baxter equation. Then we can define the cocommutator $\varphi : G[u]^\sigma \to G[u]^\sigma \otimes G[u]^\sigma$ by the formula

$$\varphi : a(u) \to [a(u) \otimes 1 + 1 \otimes a(v), r_\sigma(u,v)].$$

Cocommutator φ defines on a bisuperalgebra structure $G[u]^\sigma$.

3 Quantization

The definition of quantization for Lie bialgebras from [1] can be extended on Lie bisuperalgebras. A quantization of Lie bisuperalgebra A $(\in \{G[t], G[t]^\sigma\})$ is called a Hopf superalgebra A_h over the ring of formal power series $C[[h]]$ such that the following conditions hold:

1) $A_h/hA_h \cong U(A)$, where $U(A)$ is the universal enveloping superalgebra of a Lie superalgebra A;
2) Lie superalgebra A_h is isomorphic to $U(A)[[h]]$, as a vector space;
3) correspondence principle is fulfilled $h^{-1}(\Delta(x) - \Delta^{\mathrm{op}}(x)) \bmod h = \varphi(x \bmod h)$, where Δ is comultiplication, Δ^{op} is opposite comultiplication (i.e., if $\Delta(x) = \sum x_i' \otimes x_i''$, then $\Delta^{\mathrm{op}}(x) = \sum (-1)^{p(x_i')p(x_i'')} x_i'' \otimes x_i'$), cobracket $\varphi : A \to A \bigwedge A$ be a cocycle in A.

From 1), 2) it follows that we can consider the deformations of generators m_i, $m_i u$, $i \in I$, ($m \in \{h, x^\pm\}$ in the case of the basic Lie superalgebra and $m \in \{h, k, x^\pm\}$ in the case of Q_n), as generators of A_h. Denote these deformations as by $m_{i,0}$, $m_{i,1}$, $i \in I$, correspondingly. From conditions 1), 3) and formula for cocycle it follows that we must introduce the relations and comultiplication laws for $m_{i,0}$ as for m_i. Define a comultiplication law for $h_{i,1}$ in the case of a basic Lie superalgebra by formula:

$$\Delta(h_{i,1}) = h_{i,1} \otimes 1 + 1 \otimes h_{i,1} + h[h_{i,0} \otimes 1, \Omega_2], \qquad i \in I,$$

where $\Omega_2 = \sum\limits_{\alpha \in \Delta_+} (-1)^{\deg(x_\alpha^-)} x_\alpha^- \otimes x_\alpha^+$ is a part of Casimir operator. Let us define a comultiplication laws for $h_{i,1}$, $k_{i,1}$ in the case of Q_n by formulas

$$\Delta(h_{i,1}) = h_{i,1} \otimes 1 + 1 \otimes h_{i,1} + h[h_{i,0} \otimes 1, \Omega + id \otimes \sigma(\Omega)], \qquad i \in I,$$
$$\Delta(k_{i,1}) = k_{i,1} \otimes 1 - 1 \otimes k_{i,1} + h[1 \otimes k_{i,0}, \Omega + id \otimes \sigma(\Omega)], \qquad i \in I,$$

where $\Omega = \Omega_2 + 1/2 \sum k_i \otimes k^i$, k^i are dual to k_i. (We also denote odd generator $x_\alpha^\pm$) by $y_\alpha^\pm$).) Easy to check that the correspondence principle is fulfilled in both cases.

Comultiplication laws for $x_{i,1}^\pm$ is

$$\Delta(x_{i,1}^+) = x_{i,1}^+ \otimes 1 + (-1)^{\deg(x_i^-)} 1 \otimes x_{i,1}^+ + h[x_{i,0}^+ \otimes 1, \tilde{\Omega}], \qquad i \in I,$$
$$\Delta(x_{i,1}^-) = x_{i,1}^- \otimes 1 + (-1)^{\deg(x_i^-)} 1 \otimes x_{i,1}^- + h[1 \otimes x_{i,0}^-, \tilde{\Omega}], \qquad i \in I,$$

where $\tilde{\Omega} = \Omega_2$ in the case of the basic Lie superalgebra and $\tilde{\Omega} = \Omega + id \otimes \sigma(\Omega)$ in the case of Q_n.

From the condition of compatibility of superalgebra and cosuperalgebra structures other defining relations for A_h follow. Fixing $h = 1$ we obtain the Yangian of Lie superalgebra G: $Y(G) = A_1$.

We will use the notation $\{a, b\} := ab + (-1)^{\deg(a)\deg(b)} ba$ for anticommutator of elements a, b. Also $(b_{ij})_{i,j=1}^{n-1}$ denotes Cartan matrix for A_{n-1} Lie algebra, $\tilde{b}_{ij} = a_{n,n+i-j}$, where $(a_{ij})_{i,j=1}^{2n-1}$ be a Cartan matrix for $A(n-1, n-1)$ Lie superalgebra.

Theorem 1. 1) *Let G be a basic Lie superalgebra, $G = A(m, n)$. The Yangian $Y(G)$ is a Hopf superalgebra over C with generators $h_{i,0}$, $x_{i,0}^\pm$, $h_{i,1}$, $x_{i,1}^\pm$, $i \in I$, and defining relations*

$$[h_{i,0}, h_{j,0}] = [h_{i,0}, h_{j,1}] = [h_{i,1}, h_{j,1}] = 0, \tag{1}$$
$$[h_{i,0}, x_{j,0}^\pm] = \pm b_{ij} x_{j,0}^\pm, \qquad [h_{i,1}, x_{j,0}^\pm] = \pm b_{ij} x_{j,1}^\pm, \tag{2}$$
$$[x_{i,0}^+, x_{j,0}^-] = \delta_{ij} h_{i,0}, \qquad [x_{i,1}^+, x_{j,0}^-] = \delta_{ij} \left(h_{i,1} + \frac{1}{2} h_{i,0}^2 \right), \tag{3}$$
$$[x_{i,1}^\pm, x_{j,0}^\pm] = [x_{i,0}^\pm, x_{j,1}^\pm] \pm (b_{ij}/2)(x_{i,0}^\pm x_{j,0}^\pm + x_{j,0}^\pm x_{i,0}^\pm), \tag{4}$$
$$(\operatorname{ad} x_{i,0}^\pm)^{n_{ij}} (x_{j,0}^\pm) = 0, \qquad i \neq j, \tag{5}$$
$$[[h_{i,1}, x_{i,1}^+], x_{i,1}^-] + [x_{i,1}^+, [h_{i,1}, x_{i,1}^-]] = 0, \qquad b_{ii} \neq 0, \tag{6}$$
$$[[h_{i,1}, x_{j,1}^+], x_{j,1}^-] + [x_{j,1}^+, [h_{i,1}, x_{j,1}^-]] = 0, \qquad b_{ii} = 0, \qquad b_{ij} \neq 0, \tag{7}$$
$$[[x_{m+1,0}^\pm, x_{m+2,0}^\pm], [x_{m+1,0}^\pm, x_{m,0}^\pm]] = 0. \tag{8}$$

The comultiplication law Δ is defined by formulas:

$$
\begin{aligned}
&\Delta(h_{i,0}) = h_{i,0} \otimes 1 + 1 \otimes h_{i,0}, \qquad \Delta(x^{\pm}_{i,0}) = x^{\pm}_{1,0} \otimes 1 + (-1)^{\deg(x_i^-)} 1 \otimes x^{\pm}_{i,0},\\
&\Delta(h_{i,1}) = h_{i,1} \otimes 1 + 1 \otimes h_{i,1} + [h_{i,0} \otimes 1, \Omega_2],\\
&\Delta(x^+_{i,1}) = x^+_{i,1} \otimes 1 + (-1)^{\deg(x_i^+)} 1 \otimes x^+_{i,1} + [x^+_{i,0} \otimes 1, \Omega_2],\\
&\Delta(x^-_{i,1}) = x^-_{i,1} \otimes 1 + (-1)^{\deg(x_i^-)} 1 \otimes x^-_{i,1} - [1 \otimes x^-_{i,0}, \Omega_2].
\end{aligned}
$$

2) *The Yangian $Y(Q_n)$ is a Hopf algebra over C with generators $h_{i,0}$, $x^{\pm}_{i,0}$, $k_{i,0}$, $y^{\pm}_{i,0}$, $h_{i,1}$, $x^{\pm}_{i,1}$, $k_{i,1}$, $y^{\pm}_{i,1}$, $1 \le i \le n$, (h, x are even, k, y are odd generators). Defining relations for generators h_i, $x^{\pm}_i$ include the relations* (1), (2), (5)–(8) *and the rest defining relations is following:*

$$
\begin{aligned}
&[h_{i,0}, k_{j,0}] = [h_{i,0}, k_{j,1}] = [h_{i,1}, k_{j,1}] = 0, \qquad [k_{i,1}, k_{j,0}] = [k_{i,0}, k_{j,1}] = 0,\\
&[h_{i,0}, y^{\pm}_{j,0}] = \pm b_{ij} y^{\pm}_{j,0}, \qquad [k_{i,0}, x^{\pm}_{j,0}] = \pm b_{ij} y^{\pm}_{j,0}, \qquad [k_{i,0}, y^{\pm}_{j,0}] = \mp \tilde{b}_{ij} x^{\pm}_{j,0},\\
&[h_{i,1}, y^{\pm}_{j,0}] = \pm b_{ij} y^{\pm}_{j,1}, \qquad [k_{i,1}, x^{\pm}_{j,0}] = \pm b_{ij} y^{\pm}_{j,1}, \qquad [k_{i,1}, y^{\pm}_{j,0}] = \mp \tilde{b}_{ij} x^{\pm}_{j,1},\\
&[x^+_{i,0}, y^-_{j,0}] = [y^+_{i,0}, x^-_{j,0}] = \delta_{ij} k_{i,0},\\
&[y^+_{i,0}, y^-_{j,0}] = \delta_{ij} \left(-\frac{4}{n+1} \sum_{k=1}^{n} k h_{k,0} + 2 \sum_{k=i}^{n} h_{k,0} + 2 \sum_{k=i+1}^{n} h_{k,0} \right),\\
&[x^+_{i,1}, x^-_{j,0}] = \delta_{ij} \tilde{h}_{i,1} = \delta_{ij} \left(h_{i,1} + \frac{1}{2} (h^2_{i,0} + k^2_{i,0}) \right),\\
&[x^+_{i,1}, y^-_{j,0}] = \delta_{ij} \tilde{k}_{i,1} = \delta_{ij} \left(k_{i,1} + \frac{1}{2} \{h_{i,0}, k_{i,0}\} \right),\\
&[x^{\pm}_{i,1}, x^{\pm}_{j,0}] = [x^{\pm}_{i,0}, x^{\pm}_{j,1}] \pm (b_{ij}/2)\big(\{x^{\pm}_{i,0}, x^{\pm}_{j,0}\} + [y^{\pm}_{i,0}, y^{\pm}_{j,0}]\big),\\
&[y^{\pm}_{i,1}, y^{\pm}_{j,0}] = [y^{\pm}_{i,0}, y^{\pm}_{j,1}] \pm (\tilde{b}_{ij}/2)\{x^{\pm}_{i,0}, x^{\pm}_{j,0}\} \pm (b_{ij}/2)\{y^{\pm}_{i,0}, y^{\pm}_{j,0}\},\\
&[x^{\pm}_{i,1}, y^{\pm}_{j,0}] = [x^{\pm}_{i,0}, y^{\pm}_{j,1}] \pm (b_{ij}/2)\{x^{\pm}_{i,0}, y^{\pm}_{j,0}\} \pm (\tilde{b}_{ij}/2)\{y^{\pm}_{i,0}, x^{\pm}_{i,0}\},\\
&[[h_{i,1}, y^+_{i,1}], y^-_{i,1}] + [y^+_{i,1}, [h_{i,1}, y^-_{i,1}]] = 0, \qquad [[k_{i,1}, x^+_{i,1}], x^-_{i,1}] + [x^+_{i,1}, [k_{i,1}, x^-_{i,1}]] = 0,\\
&[[k_{i,1}, y^+_{i,1}], y^-_{i,1}] + [k_{i,1}, [y^+_{i,1}], y^-_{i,1}]] = 0,\\
&[k_{i,0}, k_{j,0}] = \frac{2}{n+1} (2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}) \sum_{k=1}^{n} h_{k,0}\\
&\qquad\qquad + 2(\delta_{i,j} - \delta_{i,j+1}) \sum_{k=i}^{n} h_{k,0} + 2(\delta_{i,j} - \delta_{i,j-1}) \sum_{k=i+1}^{n} h_{k,0},\\
&[y^+_{i,1}, y^-_{j,0}] = \delta_{ij} \left(h_{i,1} + \frac{1}{2} (h^2_{i,0} + k^2_{i,0}) \right).
\end{aligned}
$$

4 New system of generators

For simplicity we will assume that $G = A(m,n)$. Let us introduce generators $\tilde{x}^{\pm}_{i,k}$, $\tilde{h}_{i,k}$, $(\in Y(G))$, $i \in I$, $k \in Z_+$ by formulas

$$\tilde{x}^{\pm}_{i,k+1} = \pm(a^{-1}_{i,i})[h_{i,1}, \tilde{x}^{\pm}_{i,k}], \qquad i \in I \setminus \tau (i \ne m+1), \tag{9}$$

$$\tilde{x}^{\pm}_{m+1,k+1} = \mp [h_{m,1}, \tilde{x}^{\pm}_{m+1,k}], \tag{10}$$

$$\tilde{h}_{i,k} = [\tilde{x}^+_{i,k}, x^-_{i,0}], \tag{11}$$

$$\tilde{x}^{\pm}_{i,0} = x^{\pm}_{i,0}, \qquad \tilde{h}_{i,0} = h_{i,0}. \tag{12}$$

Definition 1. Denote by $\bar{Y}(G)$ a superalgebra over C with generators $x_{i,k}^{\pm}$, $h_{i,k}$, $i \in \Gamma = I$, $k \in Z_+$, and defining relations:

$$[h_{i,k}, h_{j,l}] = 0, \qquad \delta_{i,j} h_{i,k+l} = [x_{i,k}^{+}, x_{j,l}^{-}], \tag{13}$$

$$[h_{i,k+1}, x_{j,l}^{\pm}] = [h_{i,k}, x_{j,l+1}^{\pm}] + (b_{ij}/2)(h_{i,k} x_{j,l}^{\pm} + x_{j,l}^{\pm} h_{i,k}), \tag{14}$$

$$[h_{i,0}, x_{j,l}^{\pm}] = \pm b_{ij} x_{j,l}^{\pm}, \tag{15}$$

$$[x_{i,k+1}^{\pm}, x_{j,l}^{\pm}] = [x_{i,k}^{\pm}, x_{j,l+1}^{\pm}] + (b_{ij}/2)(x_{i,k}^{\pm} x_{j,l}^{\pm} + x_{j,l}^{\pm} x_{i,k}^{\pm}), \tag{16}$$

$$\sum_{\sigma} [x_{i,k_{\sigma(1)}}^{\pm}, \ldots [x_{i,k_{\sigma(r)}}^{\pm}, x_{j,l}^{\pm}] \ldots] = 0, \qquad i \neq j, \qquad r = n_{ij}. \tag{17}$$

The sum is taken over all permutations σ of $\{1, \ldots, r\}$. Let us note that the parity function takes the following values on generators: $p(x_{j,k}^{\pm}) = 0$, for $k \in Z_+$, $j \in I \setminus \tau$ $p(h_{i,k}) = 0$, for $i \in I$, $k \in Z_+$, $p(x_{i,k}^{\pm}) = 1$, $k \in Z_+$, $i \in \tau$.

Theorem 2. *Correspondence*

$$\tilde{x}_{i,k}^{\pm}(\in Y(G) \to x_{i,k}^{\pm}(\in \bar{Y}(G), \qquad \tilde{h}_{i,k}(\in Y(G) \to h_{i,k}(\in \bar{Y}(G)$$

defines the isomorphism

$$Y(G) \to \bar{Y}(G).$$

Proof. First of all I will give the sketch of the proof. We can suggest that $\bar{Y}(G)$ is generated by generators $\tilde{x}_{i,k}^{\pm}$, $\tilde{h}_{i,k}$ satisfying relations (1)–(12). Therefore, it is sufficient to prove that relations (13)–(17) follows from (1)–(12) and, again that the relations (1)–(12) follow from relations (13)–(17). The latter fact is almost evident. Really, relations (1)–(8) are contained among relations (13)–(17). The first relation (9) follows from second relation (14) and the definition of $\tilde{h}_{i,1}$. Relation (12) follows from the second relation (13) The inverse fact can be proved as in [5].

The equivalence of defining relations for even generators is similar to the case of Yangians of simple Lie algebras and may be proved by induction. The proof of equivalence of defining relations in the case of odd generators is based on the one for the even generators and is carried out the same way. First of all we deduce (13)–(17) from (1)–(8) for small values of second indices the preparing base of induction. After that we deduce relations (13)–(17) for $i = j$ (for even and odd generators) and finally we deduce (13)–(17) for $i \neq j$ for even and odd indices. ■

5 Double of Yangian and the universal *R*-matrix

Recall the definition of double of Hopf algebra (superalgebra). Let A^0 be a dual A^* with opposite comultiplication. Then the unique quasitriangular Hopf algebra (superalgebra) $D(A)$ exists with the universal R-matrix R such that: 1) $D(A)$ consists of A and A^0 as Hopf subalgebras (subsuperalgebras); 2) R is an image of the canonical element $A \otimes A^0$ under inclusion in $D(A) \otimes D(A)$; 3) the linear map $A \otimes A \hookrightarrow D(A)$, $a \otimes b \mapsto ab$ is a bijection. Let us define the double of the super Yangian. Let $C(G)$ be a superalgebra generated by the elements $x_{ik}^{\pm}$, h_{ik}, $i \in I$, $k \in Z$ with relations (13)–(17). As above, we refer to the second index of the generators as its degree. We obtain that superalgebra $C(G)$ admits Z-filtration:

$$\cdots \subset C_n \subset \cdots \subset C_{-1} \subset C_0 \subset C_1 \subset \cdots \subset C(G).$$

Let $\bar{C}(G)$ be the corresponding formal completion of $C(G)$.

Theorem 3. *Superalgebra $\bar{C}(G)$ isomorphic $DY(G)$.*

Theorem 4. *Let $G = A(m, n)$. Then the universal R-matrix R of the super Yangian double $DY(G)$ can be presented in the factorizable form*

$$R = R_+ R_0 R_-,$$

where

$$R_\pm = \prod_{\alpha\in\Delta_+,\ k\geq 0} \exp\left(N_\alpha x^{\pm}_{\alpha,k} \otimes x^{\mp}_{-\alpha,-k-1}\right),$$

where the product is ordering in correspondence with some normal (convex) order (see, for example [9]) and N_α is a some normalizing constant.

The middle term R_0 has a more complicated structure.

Theorem 5. *Let $\varphi^{\pm}_i(u) = \ln h^{\pm}_i(u)$, $A = (a_{ij})^n_{i,j=1}$ be a symmetrizable matrix for $A(m, n)$ ($m \neq n$), $a_{ij}(q) = [a_{ij}]_q = \frac{q^{a_{ij}} - q^{-a_{ij}}}{q - q^{-1}}$, $A(q) = (a_{ij}(q))^n_{i,j=1}$. Let $C(q) = (c_{ij}(q))^n_{i,j=1}$ be a proportional matrix to $A(q)^{-1}$, and T be a shift operator, $Tf(v) = f(v - 1)$. Then*

$$R_0 = \prod_{m\geq 0} \exp \sum_{i,j\in I} \mathrm{Res}_{u=v}(\varphi^+_i(u))' \otimes c_{ji}(T^{-1/2})\varphi^-_j(v + (n + 1/2)h),$$

where h is some constant, and operator $T^{-1/2}$ substitutes $c_{ji}(q)$ instead of q.

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