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On Polynomial Identities in Algebras Generated by Idempotents and Their ∗-Representations

Vyacheslav I. RABANOVICH and Alexander V. STRELETS

Institute of Mathematics of NAS Ukraine, 3 Tereschenkivska Str., 01601 Kyiv-4, Ukraine
E-mail: *slavik@imath.kiev.ua, sav@imath.kiev.ua*

Among algebras generated by linearly dependent idempotents we look for algebras with polynomial identities. One of the way to find a polynomial identity in an algebra is to calculate a linear basis for this algebra, build a "big" family of representations and prove using the basis that it is a residual family. In the paper was found a linear basis for some algebra generated by idempotents.

1 Introduction

Algebras generated by linearly dependent idempotents are investigated in the paper. Among these algebras we look for algebras with polynomial identities, so called PI-algebras (see, for example, [1]). The theory of PI-algebras is well developed and gives additional information about algebras. So for applications it is important to determine that an algebra has a polynomial identity. One of the way to find a polynomial identity in an algebra is to calculate a linear basis for this algebra, build a "big" family of representations such that the supremum of dimensions of representations from the family is finite and prove that it is a residual family (see [2, Theorem 2] or [3]).

Algebras which we are interested in are

$$\mathbf{Q}_{n,\overline{\lambda}} = \mathbb{C}\langle q_1, \ldots, q_n \,|\, q_k^2 = q_k, \sum_{k=1}^{n} \lambda_k q_k = e\rangle,$$

$n \in \mathbb{N}$, $\overline{\lambda} \in \mathbb{C}^n$, and its factor-algebras $\mathbf{Q}_{n,\overline{\lambda}}/\{q_{i_l} q_{j_l} = q_{j_l} q_{i_l} = 0\}_{l=1}^{m}$ where $i_l \neq j_l \in \{1, \ldots, n\}$.

In paper [4] a criterion was given when algebras $\mathbf{Q}_{n,\overline{\lambda}}$ are PI-algebras (note that the case $\lambda_i = \frac{1}{\lambda}$ was studied in the paper [5]). We remind the respective results.

In the case $n \leqslant 3$ all algebras $\mathbf{Q}_{n,\overline{\lambda}}$ are finite-dimensional and so they are PI-algebras. In the case $n \geqslant 4$ all algebras $\mathbf{Q}_{n,\overline{\lambda}}$ are infinite-dimensional. Let $\hat{\delta}(\overline{\lambda}) = 1 - \frac{1}{2}\sum_{j=1}^{n} \lambda_j$. Then we have the following results.

Theorem 1. *If $\hat{\delta}(\overline{\lambda}) \neq 0$ then the algebra $\mathbf{Q}_{4,\overline{\lambda}}$ is not a PI-algebra.*

Corollary 1. *When $n \geqslant 5$ all algebras $\mathbf{Q}_{n,\overline{\lambda}}$ are not PI-algebras.*

Theorem 2. *If $\hat{\delta}(\overline{\lambda}) = 0$ then the algebra $\mathbf{Q}_{4,\overline{\lambda}}$ is an F_4-algebra, i.e.*

$$\sum_{\sigma \in S_4} (-1)^{p(\sigma)} v_{\sigma(1)} v_{\sigma(2)} v_{\sigma(3)} v_{\sigma(4)} = 0,$$

for any elements $v_1, v_2, v_3, v_4 \in \mathbf{Q}_{n,\overline{\lambda}}$, where S_4 is a symmetric group.

The Corollary 1 shows that when $n \geqslant 5$ algebras $\mathbf{Q}_{n,\overline{\lambda}}$ are not PI-algebras. But if we add some relations, for example, relations of commuting ($q_iq_j = q_jq_i$) or orthogonality ($q_iq_j = 0$) for some generators, then some of obtained factor-algebras are PI-algebras and are infinite-dimensional.

In paper [6] for the algebra

$$\mathcal{R} = \mathbb{C}\langle p_1, p_2, p_3, q_1, q_2, q_3 \,|\, q_k^2 = q_k, p_k^2 = p_k, p_kq_k = 0, \sum_{k=1}^{3}(p_k + 2q_k) = 3e\rangle,$$

the following results were obtained:

Lemma 1. *The algebra $\mathcal{R}$ is infinite-dimensional and has the quadratic growth.*

Theorem 3. *The algebra $\mathcal{R}$ is an F_6-algebra.*

In both cases to prove Theorem 2 and Theorem 3 a residual family for corresponding algebras was found.

In this paper we will find a linear basis of an algebra

$$\mathbf{A} = \mathbb{C}\langle a, b \,|\, a^3 + \alpha_1 a + \alpha_0 = 0,\ b^3 + \beta_1 a + \beta_0 = 0,\ (a+b)^3 + \gamma_1(a+b) + \gamma_0 = 0\rangle,$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$. But first we give some argumentation to solve this problem. Consider algebras

$$\mathbf{A}' = \mathbb{C}\langle p_1, p_2, p_3, q_1, q_2, q_3 \,|\, q_k^2 = q_k, p_k^2 = p_k, p_kq_k = 0, \sum_{k=1}^{3}(\mu_kp_k + \nu_kq_k) = e\rangle,$$

where $\mu_k, \nu_k \in \mathbb{C}$, $\mu_k \neq \nu_k$, $\sum\limits_{k=1}^{3}(\mu_k + \nu_k) = 3$ (in the case of $\mu_k = 1/3$, $\nu_k = 2/3$ we have the algebra $\mathcal{R}$). To find an answer the question whether some family of representations is a residual family we need a linear basis in this algebra or in some algebra which is isomorphic with this one.

The algebra $\mathbf{A}'$ and an algebra

$$\mathbf{A}'' = \mathbb{C}\langle x_1, x_2, x_3 \,|\, x_1 + x_2 + x_3 = e,\ x_k(x_k - \mu_k)(x_k - \nu_k) = 0\rangle$$

are isomorphic. A map $x_k \mapsto \mu_kp_k + \nu_kq_k$ is a corresponding isomorphism. A map $x_k \mapsto x'_k + (\mu_k + \nu_k)/3$ gives isomorphism between the algebra $\mathbf{A}''$ and an algebra

$$\mathbf{A}''' = \mathbb{C}\langle x'_1, x'_2, x'_3 \,|\, x'_1 + x'_2 + x'_3 = 0,\ f_k(x_k) = 0\rangle,$$

where f_k are polynomials such that $\deg f_k = 3$ and sum of roots is zero. So this algebra and the algebra $\mathbf{A}$ are isomorphic.

2 A linear basis in the algebra A

We introduce a homogeneous lexicographical order on words in alphabet $\{a, b\}$: $a < b$. Using the Diamond Lemma (for definitions of the notions *Gröbner basis, reductions, compositions, growth of an algebra* etc. see review [7] and references therein) we will prove the following theorem which describes a linear basis in the algebra $\mathbf{A}$. We will denote reductions by symbol $\to$.

Theorem 4. *The set of words*

$$\{a^{\sigma_1}(ba^2)^n(ba)^m b^{\sigma_2} \,|\, \sigma_1, \sigma_2 \in \{0,1,2\}, n, m \in \mathbb{N} \cup \{0\}\}$$

is a linear basis of the algebra $\mathbf{A}$.

Proof. Let us introduce notations

$$\begin{aligned}
&p = -\alpha_1 a - \alpha_0,\\
&q = -\beta_1 b - \beta_0,\\
&s = \gamma_1(a+b) + \gamma_0,\\
&r = s + p + q,\\
&\Lambda = bab + ba^2 + ab^2 + aba + a^2b.
\end{aligned}$$

Then $(a+b)^3 + \gamma_1(a+b) + \gamma_0 \to b^2a + \Lambda + r$. So we have an ideal I generated by elements $a^3 - p$, $b^3 - q$, $b^2a + \Lambda + r$. Let G be the set of this elements. The main words of the set G are b^3, b^2a and a^3. So we have 7 compositions:

$$b\cdot b^2\cdot b,\ b^2\cdot b\cdot b^2,\ b\cdot b^2\cdot a,\ b^2\cdot b\cdot ba,\ a\cdot a^2\cdot a,\ a^2\cdot a\cdot a^2 \text{ and } b\cdot ba\cdot a^2.$$

The first composition is a sub-word of the second, the third is a sub-word of the fourth and the fifth is a sub-word of the sixth, so by the Triangle Lemma (see for example [7, sec. 2.10]) it is enough to calculate only the first, the third, the fifth and the seventh compositions. But the first and the fifth compositions are reduced to 0 because $[b,q] = 0$ and $[a,p] = 0$.

By reductions of the third composition we get 0:

$$\begin{aligned}
b\cdot b^2\cdot a:\ \ b\Lambda + br + qa^2 &= \underline{\underline{b^2ab}} + \underline{b^2a^2} + \underline{\underline{bab^2}} + \underline{baba} + \underline{\underline{ba^2b}} + br + qa\\
&= (b^2a + bab + ba^2)b + (b^2a + bab)a + br + qa\\
&\to -(ab^2 + \underline{aba} + \underline{a^2b} + r)b - (ba^2 + \underline{ab^2} + \underline{aba} + \underline{a^2b} + r)a + br + qa\\
&\to -a(b^2a + bab + ba^2 + ab^2 + aba) - bp + [q,a] + [b,r] - ra\\
&\to a(a^2b + r) - bp + [q,a] + [b,r] - ra\\
&\to [b+a,r] - [a,q] - [b,p] = [b+a,p+q] - [a,q] - [b,p] = 0.
\end{aligned}$$

The seventh composition gives a new element:

$$\begin{aligned}
b^2\cdot a\cdot a^2:\ \ \Lambda a^2 + ra^2 + b^2p &\to baba^2 + \underline{ab^2a^2} + \underline{a^2ba^2} + bap + abp + ra^2 + b^2p\\
&= baba^2 + a(b^2a + aba)a + \{bp, a\} + ra^2 + b^2p\\
&\to baba^2 - ababa - abp - a^2b^2a - pba - ara + \{bp,a\} + ra^2 + b^2p\\
&\to baba^2 - ababa + a^2(\Lambda + r) - ara + [ba,p] + ra^2 + b^2p\\
&\to baba^2 - ababa + a^2bab + a^2ba^2\\
&\quad + pb^2 + pba + pab + a^2r - ara + [ba,p] + ra^2 + b^2p\\
&= baba^2 - ababa + a^2bab + a^2ba^2 + \{b^2,p\} + \{b,ap\} + \{a^2,r\} - ara.
\end{aligned}$$

We introduce notations $\Sigma = \{a^2, r\} - ara$ and $\Omega = \{b^2,p\} + \{b,ap\}$ and add the element

$$baba^2 - ababa + a^2bab + a^2ba^2 + \Omega + \Sigma$$

to the set G.

So we obtain new compositions: $b\cdot ba\cdot ba^2$, $b^2\cdot ba\cdot ba^2$, $bab\cdot a^2\cdot a$ and $baba\cdot a\cdot a^2$. Again the first composition is a sub-word of the second and the third composition is a sub-word of the forth. And we need to calculate only first and third ones.

To calculate the first composition we use that

$$\Omega = \{b^2,p\} + \{b,ap\} = \alpha_1(\{b^2,a\} + \{b,a^2\}) + 2\alpha_0(b^2 + ba + ab)$$

$$\to -\alpha_1(bab + aba + r) + \alpha_0(2b^2 + ba + ab).$$

Thus we have

$$\begin{aligned}
bab\cdot a^2\cdot a:\ & -\underline{ababa^2} + \underline{a^2baba} + a^2bp + \Omega a + \Sigma a + babp\\
&= -a(baba^2 - ababa) + a^2bp + \Omega a + \Sigma a + babp\\
&\to a(a^2bab + a^2ba^2 + \Omega + \Sigma) + a^2bp + \Omega a + \Sigma a + babp\\
&\to pba^2 + pbab + a^2bp + babp + \{\Omega, a\} + \{\Sigma, a\}\\
&= \alpha_1(aba^2 + abab + a^2ba + baba - \{bab + aba + r, a\})\\
&+ \alpha_0(ba^2 + 2bab + a^2b + \{2b^2 + ba + ab, a\}) + \{a^2r + ra^2 - ara, a\}\\
&\to -\alpha_1\{r, a\} - 2\alpha_0(b^2a + \Lambda) + a^2ra + rp - ara^2 + pr + ara^2 - a^2ra\\
&\to -\alpha_1\{r, a\} - 2\alpha_0 r + \{r, p\} = 0.
\end{aligned}$$

The third composition also does not give a new element into the set G:

$$\begin{aligned}
b\cdot ba\cdot ba^2:\ & (\underline{\underline{bab}} + \underline{\underline{ba^2}} + ab^2 + aba + a^2b + r)ba^2 - b(-\underline{ababa} + \underline{a^2bab} + \underline{a^2ba^2} + \Omega + \Sigma)\\
&= ba(b^2a + bab + ab^2 + aba + a^2b)a + ababa^2 + a^2b^2a^2\\
&- ba^2(b^2a + bab + ba^2 + aba) + (aqa^2 + rba^2 - b\Omega - b\Sigma)\\
&\to -ba(\underline{\underline{ba^2}} + r)a + \underline{\underline{ababa^2}} + a^2b^2a^2 + ba^2(ab^2 + a^2b + r)\\
&+ (aqa^2 + rba^2 - b\Omega - b\Sigma)\\
&\to -(baba^2 - ababa)a + a^2b^2a^2\\
&+ (aqa^2 + rba^2 - b\Omega - b\Sigma - bara + bpb^2 + bpab + ba^2r)\\
&\to (\underline{\underline{a^2bab}} + \underline{\underline{a^2ba^2}} + \Omega + \Sigma)a + \underline{\underline{a^2b^2a^2}}\\
&+ (aqa^2 + rba^2 - b\Omega - b\Sigma - bara + bpb^2 + bpab + ba^2r)\\
&= a^2(b^2a + bab + ba^2)a\\
&+ (aqa^2 + rba^2 - b\Omega - b\Sigma - bara + bpb^2 + bpab + ba^2r + \Omega a + \Sigma a)\\
&\to -a^2(ab^2 + aba + a^2b + r)a\\
&+ (aqa^2 + rba^2 - b\Omega - b\Sigma - bara + bpb^2 + bpab + ba^2r + \Omega a + \Sigma a)\\
&\to -p(b^2a + ba^2 + aba)a\\
&+ (aqa^2 + rba^2 - b\Omega - b\Sigma - bara + bpb^2 + bpab + ba^2r + \Omega a + \Sigma a - a^2ra)\\
&\to aqa^2 + rba^2 - \underline{\underline{b\Omega}} - \underline{b\Sigma} - bara + bpb^2 + bpab + ba^2r + \underline{\underline{\Omega a}} + \underline{\Sigma a} - a^2ra\\
&+ pbab + pab^2 + pa^2b + pr\\
&= aqa^2 + rba^2 - bara + bpb^2 + bpab + ba^2r - a^2ra + \underline{\underline{pbab}} + \underline{\underline{pab^2}} + \underline{\underline{pa^2b}} + \underline{\underline{pr}}\\
&+ (b^2pa + \underline{\underline{pb^2a}} + bapa + \underline{\underline{apba}} - qp - bpb^2 - b^2ap - bapb)\\
&+ (a^2ra + rp - ara^2 - ba^2r - bra^2 + bara)\\
&= p(b^2a + bab + ab^2 + aba + a^2b + r)\\
&+ aqa^2 + rba^2 + bapa - qp + rp - ara^2 - bra^2\\
&\to -pba^2 + aqa^2 + rba^2 + bapa - qp + rp - ara^2 - bra^2\\
&= ([b, p] + [a, q] + [r, b])a^2 + qa^3 - qp + rp - ara^2\\
&\to ([s, b] + [s, a] - [q, a] - [s, a])a^2 - ara^2 + rp = -([r, a] + ar)a^2 + rp \to 0.
\end{aligned}$$

Then by the Diamond Lemma the main words of the Gröbner bases G are b^3, a^3, b^2a, $baba^2$, so they are disallowed and the theorem is proved. ■

Corollary 2. *Algebras* **A** *are infinite-dimensional and have the quadratic growth.*

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