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New Frobenius Structures on Hurwitz Spaces in Terms of Schiffer and Bergmann Kernels

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New family of flat potential (Darboux–Egoroff) metrics on the Hurwitz spaces and corresponding Frobenius structures are found. We consider a Hurwitz space as a real manifold. Therefore the number of coordinates is twice as big as the number of coordinates used in the construction of Frobenius structure on Hurwitz spaces found by B. Dubrovin more than 10 years ago. The branch points of a ramified covering and their complex conjugates play the role of canonical coordinates on the constructed Frobenius manifolds. We introduce a new family of Darboux–Egoroff metrics in terms of the Schiffer and Bergmann kernels, find corresponding flat coordinates and a prepotential of associated Frobenius manifolds.

1 Introduction

The aim of the present work is to construct a new class of Frobenius manifolds associated to the Hurwitz spaces. First, we describe Dubrovin's structure of Frobenius manifolds on Hurwitz spaces. Those manifolds have complex dimension L, where L is complex dimension of the Hurwitz space. In our approach we consider the Hurwitz space as a real manifold, therefore the constructed here Frobenius manifolds have $2L$ coordinates.

We start with construction of a new family of flat diagonal metrics on the Hurwitz space written in terms of the Schiffer and Bergmann reproducing kernels on a ramified covering of Riemann sphere with simple branch points. Each of these metrics gives a Frobenius algebra on the tangent space. Then we find flat coordinates of the metrics and prepotentials F of the Frobenius structures. The prepotential F satisfies the WDVV system with respect to the flat coordinates for the corresponding metric according to the one-to-one correspondence between Frobenius structures and solutions to the WDVV equations [1].

2 Frobenius manifolds and WDVV equations

The theory of Frobenius manifolds is related to various branches of mathematics such as singularity theory, reflection groups, algebraic geometry, isomonodromic deformation theory, boundary-value problems, and Painlevé equations. It was developed by B. Dubrovin as a geometric interpretation of the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations of the following type:

$$F_i F_1^{-1} F_j = F_j F_1^{-1} F_i, \qquad i, j = 1, \dots, n,$$

where F_i is the matrix

$$(F_i)_{mn} = \frac{\partial^3 F}{\partial t^i \partial t^m \partial t^n}$$

for the function F of n variables $t^1, \dots, t^n$ with the normalization: F_1 is a constant nondegenerate matrix.

Any solution $F(t)$, $t = (t^1, \dots, t^n) \in M$ of these WDVV equations which satisfies an additional assumption of quasihomogenuity,

$$F(c^{d_1}t^1, \dots, c^{d_n}t^n) = c^{d_F}F(t^1, \dots, t^n) \qquad \text{for some } c \neq 0 \text{ and } d_1, \dots, d_n, d_F \in \mathbb{R},$$

determines in the domain M the structure of Frobenius manifold and vice versa (see [1]).

Definition 1. An algebra A over $\mathbb{C}$ is called (commutative) Frobenius algebra if:

- it is a commutative associative $\mathbb{C}$-algebra with a unity e.
- it is supplied with a $\mathbb{C}$-bilinear symmetric nondegenerate inner product having the property $\langle a \cdot b, c\rangle = \langle a, b \cdot c\rangle$

Definition 2. M is Frobenius manifold if a structure of Frobenius algebra is specified on any tangent plane T_tM at any point $t \in M$ smoothly depending on the point such that

1) the inner product $\langle , \rangle$ is a flat metric on M,
2) the unity vector field e is covariantly constant w.r.t. the Levi-Civita connection ∇ for the metric $\langle , \rangle$: $\nabla e = 0$,
3) the tensor $(\nabla_z \boldsymbol{c})(u, v, w)$ is symmetric in four vector fields u, v, w, z, where $\boldsymbol{c}$ is the following symmetric 3-tensor: $\boldsymbol{c}(u, v, w) = \langle u \cdot v, w\rangle$,
4) a vector field E is determined on M such that $\nabla(\nabla E) = 0$,

$$\begin{aligned} &[E, x \cdot y] - [E, x] \cdot y - x \cdot [E, y] = x \cdot y, \\ &E\langle x, y\rangle - \langle [E, x], y\rangle - \langle x, [E, y]\rangle = (2 - \nu)\langle x, y\rangle. \end{aligned} \tag{1}$$

for some constant ν, the charge of Frobenius manifold.

We take the local coordinates $\lambda_1, \dots, \lambda_L$ on the Hurwitz space to be canonical coordinates for multiplication, i.e. $\partial_i \cdot \partial_j = \delta_{ij}\partial_i$, where $\partial_i = \partial_{\lambda_i}$. Then, the unit vector field is $e = \sum_{i=1}^{L} \partial_i$, and the Euler vector field is given by $E = \sum_{i=1}^{L} \lambda_i\partial_i$. Then in these coordinates the metric is diagonal: $d\boldsymbol{s}^2 = \sum_{i=1}^{L} \eta_{ii} d\lambda_i^2$, where η_{ii} are functions of $\{\lambda_i\}$. The rotation coefficients of the metric are defined as

$$\beta_{ij} = \frac{\partial_j \sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}}, \qquad i \neq j.$$

The flatness of a potential metric is provided by the system

$$\partial_k \beta_{ij} = \beta_{ik}\beta_{kj} \qquad \text{for different } i, j, k, \tag{2}$$

$$\sum_k \partial_k \beta_{ij} = 0. \tag{3}$$

(The metric is potential if and only if all rotation coefficients are symmetric with respect to indices: $\beta_{ij} = \beta_{ji}$.) The existence of the Euler vector field on the manifold (see the definition of Frobenius manifolds) implies another relation on rotation coefficients:

$$\sum_k \lambda_k \partial_k \beta_{ij} = -\beta_{ij}. \tag{4}$$

The definition of Frobenius manifold implies the existence of the function F such that the 3-tensor c is given by its third derivatives

$$\boldsymbol{c}(\partial_{t^A}, \partial_{t^B}, \partial_{t^C}) = \frac{\partial^3 F}{\partial t^A \partial t^B \partial t^C},$$

where $\{t^A\}$ is the set of flat coordinates for the metric. This function F is called the prepotential of the Frobenius manifold.

3 Hurwitz spaces

Hurwitz space is the moduli space of the pairs $(\mathcal{L}, \lambda)$, where $\mathcal{L}$ is a compact Riemann surface of genus g and λ is a meromorphic function on $\mathcal{L}$ of degree N. This function λ realizes the surface $\mathcal{L}$ as an N-fold ramified covering of $\mathbb{C}P^1$. We consider the space $M = M_{g;n_0,\dots,n_m}$, which is the moduli space of the genus g coverings of $\mathbb{C}P^1$ with fixed type of ramification over the point at infinity: $n_0, \dots, n_m \in \mathbb{N}$, $\sum_{k=0}^{m}(n_k+1) = N$.

Denote by $\lambda_1, \dots, \lambda_L$ the finite critical values of the function λ (the branch points of the covering): $\lambda_j = \lambda(P_j)$, $d\lambda|_{P_j} = 0$, $j = 1, \dots, L$, and consider them as local coordinates on M. We assume that all the branch points λ_j are simple.

4 Kernels on Riemann surfaces

1. The kernel $W(P,Q)$ defined by $W(P,Q) = \mathrm{d}_P \mathrm{d}_Q \log E(P,Q)$ is the unique symmetric differential on $\mathcal{L}\times\mathcal{L}$ with the second order pole at $P = Q$ and properties:

$$\oint_{a_j} W(P,Q) = 0, \qquad \oint_{b_j} W(P,Q) = 2\pi i \omega_j(P), \qquad j = 1, \dots, g,$$

where $\{a_j, b_j\}_{j=1}^{g}$ is the canonical basis of cycles on $\mathcal{L}$; $\{\omega_j(P)\}_{j=1}^{g}$ is the corresponding set of holomorphic differentials normalized by $\oint_{a_i}\omega_j = \delta_{ij}$; and $E(P,Q)$ is the prime form on the surface $\mathcal{L}$.

Variational formulas for $W(P,Q)$ [2,3]:

$$\frac{dW(P,Q)}{d\lambda_k} = \frac{1}{2}\frac{W(P,P_k)}{dx_k}\frac{W(Q,P_k)}{dx_k},$$

$x_k = \sqrt{\lambda - \lambda_k}$ is the local parameter near ramification point P_k and the notation $W(P,P_k)/dx_k$ should be understood as $W(P,P_k)/dx_k = (W(P,Q)/dx(Q))\,|_{Q=P_k}$.

2. The Schiffer kernel $\Omega(P,Q)$ is the symmetric differential on $\mathcal{L}\times\mathcal{L}$ [4] defined as

$$\Omega(P,Q) = W(P,Q) - \pi\sum_{i,j=1}^{g}(\operatorname{Im}\mathbb{B})^{-1}_{ij}\omega_i(P)\omega_j(Q).$$

3. The Bergmann kernel $B(P,\bar{Q})$ is the following differential on $\mathcal{L}\times\mathcal{L}$ [4]:

$$B(P,\bar{Q}) = \pi\sum_{i,j=1}^{g}(\operatorname{Im}\mathbb{B})^{-1}_{ij}\omega_i(P)\overline{\omega_j(Q)},$$

where $\mathbb{B}$ is the symmetric matrix of b-periods of ω_j: $\mathbb{B}_{ij} = \oint_{b_i}\omega_j$ ($\operatorname{Im}\mathbb{B}$ is positive definite). The Bergmann and Schiffer kernels have the following properties:

$$\oint_{a_i}\Omega(P,Q) = -\oint_{a_i}B(\bar{P},Q), \qquad \oint_{b_i}\Omega(P,Q) = -\oint_{b_i}B(\bar{P},Q),$$

where integration is taken with respect to the first argument.

Variational formulas for the Schiffer and Bergmann kernels:

$$\frac{d\Omega(P,Q)}{d\lambda_k} = \frac{1}{2}\frac{\Omega(P,P_k)}{dx_k}\frac{\Omega(Q,P_k)}{dx_k}, \qquad \frac{d\Omega(P,Q)}{d\bar{\lambda}_k} = \frac{1}{2}\frac{B(P,\bar{P}_k)}{\overline{dx_k}}\frac{B(Q,\bar{P}_k)}{\overline{dx_k}},$$
$$\frac{dB(P,\bar{Q})}{d\lambda_k} = \frac{1}{2}\frac{\Omega(P,P_k)}{dx_k}\frac{B(\bar{Q},P_k)}{dx_k}, \qquad \frac{dB(P,\bar{Q})}{d\bar{\lambda}_k} = \frac{1}{2}\frac{B(P,\bar{P}_k)}{\overline{dx_k}}\overline{\frac{\Omega(Q,P_k)}{dx_k}}.$$

Here as before $\Omega(P,P_k)/dx_k$ stands for $(\Omega(P,Q)/dx_k(Q))\,|_{Q=P_k}$.

To prove these formulas one uses the Rauch variational formulas for the holomorphic normalized differentials and for the matrix of b-periods [2]:

$$\frac{d\omega_j(P)}{d\lambda_k} = \frac{1}{2}\omega_j(P_k)W(P,P_k), \qquad \frac{d\mathbb{B}_{ij}}{d\lambda_k} = \pi i\, \omega_i(P_k)\omega_j(P_k),$$

and the variational formula for the kernel $W(P,Q)$ given above.

5 Dubrovin's Frobenius structure on Hurwitz spaces

We start with a reformulation of Dubrovin's structure of the Frobenius manifold on moduli space $M = M_{g;n_0,\dots,n_m}$ of coverings $(\mathcal{L},\lambda)$ of genus $g \geq 1$ using the kernel $W(P,Q)$. The branch points $\lambda_1,\dots,\lambda_L$ are used as local coordinates on the space M.

The flat metric on M. For any two tangent vectors ∂', ∂'' consider the inner product $\langle\partial',\partial''\rangle_\phi = \Omega_{\phi^2}(\partial'\cdot\partial'')$, where

- ϕ is one of the primary differentials listed below;
- Ω_{ϕ^2} is a one form on M: $\Omega_{\phi^2} = \sum\limits_{i=1}^{L}\left(\mathrm{res}_{P_i}\frac{\phi^2}{d\lambda}\right)d\lambda_i$;
- multiplication of the tangent vector fields is defined as follows: $\partial_i\cdot\partial_j = \delta_{ij}\partial_i$ for $\partial_i = \partial/\partial\lambda_i$.

Primary differentials. (z_i is a local parameter near ∞^i ; $z_i^{-n_i-1} = \lambda$.)

Type1. $\phi_{t^{i;\alpha}}(P) = \frac{1}{\alpha}\mathrm{res}_{Q=\infty^i} z_i^{-\alpha}(Q)W(P,Q) \sim z_i^{-\alpha-1}dz_i, \quad P\sim\infty^i,$
$i = 0,\dots,m; \quad \alpha = 1,\dots,n_i.$

Type2. $\phi_{v^i}(P) = \mathrm{res}_{Q=\infty^i}\lambda(Q)W(P,Q) \sim -d\lambda, \quad P\sim\infty^i; \quad i=1,\dots,m.$

Type3. $\phi_{w^i}(P) = \mathrm{v.p.}\int_{\infty^0}^{\infty^i} W(P,Q), \quad \mathrm{res}_{\infty^i}\phi_{w^i} = 1; \quad \mathrm{res}_{\infty^0}\phi_{w^i} = -1; \quad i=1,\dots,m.$

Type4. $\phi_{r^i}(P) = -\oint_{a_i}\lambda(Q)W(P,Q), \quad \phi_{r^i}(P+b_i) = \phi_{r^i}(P) + 2\pi i d\lambda; \quad i=1,\dots,g.$

Type5. $\phi_{s^i}(P) = \frac{1}{2\pi i}\oint_{b_i} W(P,Q), \quad \oint_{a_i}\phi_{s^i} = 1; \quad i=1,\dots,g.$

Note that all above differentials have zero a-periods except for $\oint_{a_j}\phi_{s^i} = \delta_{ij}$.

Theorem 1. *With $\phi = \phi_{t^{i;\alpha}}$; $\phi = \sum\limits_{i=1}^{m}\delta_i\phi_{v^i}$; $\phi = \sum\limits_{i=1}^{m}\delta_i\phi_{w^i}$; $\phi = \sum\limits_{i=1}^{g}\delta_i\phi_{r^i}$; $\phi = \sum\limits_{i=1}^{g}\delta_i\phi_{s^i}$ the metric $\langle,\rangle_\phi$ is flat and its rotation coefficients are given by the kernel $W(P,Q)$ at the branch points* [3]*:*

$$\beta_{ij} = \frac{1}{2}\frac{W(P_i,P_j)}{dx_i\, dx_j}.$$

The rotation coefficients satisfy equations (2)–(4)*. Requirement* (1) *holds for the metric.*

Flat coordinates for the metric $\langle,\rangle_\phi$. (I.e. the coordinates with respect to which the metric is constant.)

$$t^{i;\alpha} = \mathrm{res}_{P=\infty^i} z_i^{\alpha}(\mathrm{v.p.}\int_{\infty^0}^{P}\phi)d\lambda, \qquad i=0,\dots,m; \quad \alpha=1,\dots,n_i,$$

$$v^i = -\mathrm{v.p.}\int_{\infty^0}^{\infty^i}\phi, \qquad w^i = -\mathrm{res}_{\infty^i}\lambda\phi, \qquad i=1,\dots,m,$$

$$r^k = \frac{1}{2\pi i}\oint_{b_k}\phi, \qquad s^k = -\oint_{a_k}\lambda\phi, \qquad k=1,\dots,g.$$

Denote the flat coordinates by t^A, i.e., let $t^A \in \{t^{i;\alpha}; v^i, w^i; r^k, s^k \,|\, i=0,\dots,m,\ \alpha=1,\dots,n_i;$ $k=1,\dots,g\}$.

Prepotential of the Frobenius structure is a function $F(t^A)$ the third derivatives of which are given by the 3-tensor $\mathbf{c}$ from the definition of Frobenius manifold:

$$\frac{\partial^3 F(t)}{\partial_{t^A}\partial_{t^B}\partial_{t^C}} = \mathbf{c}(\partial_{t^A}, \partial_{t^B}, \partial_{t^C}) = \langle \partial_{t^A}\cdot\partial_{t^B}, \partial_{t^C}\rangle_\phi.$$

Let $\omega^{(1)}$ and $\omega^{(2)}$ be two differentials on $\mathcal{L}$ holomorphic outside of infinity with the following behaviour at infinities:

$$\omega^{(i)} = \sum_{k=-k_1}^{\infty} c_{k,a}^{(i)} z_a^k dz_a + \frac{1}{n_a+1} d\left(\sum_{k>0} r_{k,a}^{(i)}\lambda^k \log\lambda\right), \qquad P\sim\infty^a,$$

where $k_1\in\mathbb{Z}$ and $c_{k,a}^{(i)}$, $r_{k,a}^{(i)}$ are some coefficients. Denote also for $k=1,\dots,g$:

$$\oint_{a_k}\omega^{(i)} = A_k^{(i)},$$

$$\omega^{(i)}(P^{a_k}) - \omega^{(i)}(P) = dp_k^{(i)}(\lambda), \qquad p_k^{(i)}(\lambda) = \sum_{s>0} p_{sk}^{(i)}\lambda^s,$$

$$\omega^{(i)}(P^{b_k}) - \omega^{(i)}(P) = dq_k^{(i)}(\lambda), \qquad q_k^{(i)}(\lambda) = \sum_{s>0} q_{sk}^{(i)}\lambda^s.$$

Here $\omega(P^{a_k})-\omega(P)$ denotes an additive transformation of differential under analytic continuation along the cycle a_k on the Riemann surface.

Definition 3. For two such differentials define a pairing:

$$\langle\omega^{(i)}\ \omega^{(j)}\rangle = \sum_{a=0}^{m}\left[\sum_{n\geq 0}\frac{c_{-n-2,a}^{(i)}}{n+1}c_{n,a}^{(j)} + c_{-1,a}^{(i)}\mathrm{v.p.}\int_{P_0}^{\infty^a}\omega^{(j)} - \mathrm{v.p.}\int_{P_0}^{\infty^a} r_{n,a}^{(i)}\lambda^n\omega^{(j)}\right]$$
$$+\frac{1}{2\pi i}\sum_{k=1}^{g}\left[-\oint_{a_k} q_k^{(i)}(\lambda)\omega^{(j)} + \oint_{b_k} p_k^{(i)}(\lambda)\omega^{(j)} + A_k^{(i)}\oint_{b_k}\omega^{(j)}\right].$$

P_0 is the point on the curve $\mathcal{L}$ such that $\lambda(P_0)=0$; all basic cycles $\{a_k\}$ and $\{b_k\}$ start at P_0.

Consider the multivalued function on $\mathcal{L}$: $p(P) = \mathrm{v.p.}\int_{\infty^0}^{P}\phi$. The differential $pd\lambda$ is of the above type.

Theorem 2. *The prepotential of the Frobenius structure is* $F=\frac{1}{2}\langle pd\lambda\ pd\lambda\rangle$.

Theorem 3. *For the second derivatives of the prepotential we have* $\partial_{t^A}\partial_{t^B}F = \langle\phi_{t^A}\ \phi_{t^B}\rangle$.

6 New Frobenius structure on Hurwitz spaces

We consider the space $M = M_{g;n_0,\dots,n_m}$ as a real manifold with the coordinates $\{\lambda_1,\dots,\lambda_L;$ $\bar\lambda_1,\dots,\bar\lambda_L\}$. On this space we introduce a Frobenius structure analogous to the one described in Section 5. The new Darboux–Egoroff metric, corresponding flat coordinates and primary differentials are written in terms of the Schiffer and Bergmann kernels which depend on both $\{\lambda_k\}$ and $\{\bar\lambda_k\}$.

The flat metric on M. For two tangent vectors ∂', ∂'' consider the inner product $\langle\partial',\partial''\rangle_{\phi+\psi} = \Omega_{\phi^2+\psi^2}(\partial'\cdot\partial'')$, where

- ϕ is "holomorphic" and ψ is "antiholomorphic" parts of one of the primary differentials listed below;
- $\Omega_{\phi^2+\psi^2}$ is a one form on M: $\Omega_{\phi^2+\psi^2} = \sum_{i=1}^{L} \left(\mathrm{res}_{P_i} \frac{\phi^2}{d\lambda}(d\lambda_i) + \tilde{\mathrm{res}}_{P_i} \frac{\psi^2}{d\bar\lambda}(d\bar\lambda_i) \right)$, where we denote $\tilde{\mathrm{res}} f = \overline{\mathrm{res} \bar f}$;
- multiplication of the tangent vector fields is defined as follows: $\partial_i \cdot \partial_j = \delta_{ij}\partial_i$, where i and j run through the index sets $\{k\}_{k=1}^{L}$ and $\{\bar k\}_{k=1}^{L}$, and $\partial_i = \partial/\partial\lambda_i$, $\partial_{\bar i} = \partial/\partial\bar\lambda_i$.

Primary differentials. (z_i is a local parameter near ∞^i; $z_i^{-n_i-1} = \lambda$.)

Type 1. $i = 0, \ldots, m$; $\alpha = 1, \ldots, n_i$.

a) $\Phi_{t^{i;\alpha}}(P) = \phi_{t^{i;\alpha}}(P) + \psi_{t^{i;\alpha}}(P) = \dfrac{1}{\alpha}\mathrm{res}_{Q=\infty^i} z_i^{-\alpha}(Q)\Omega(P,Q) + \dfrac{1}{\alpha}\mathrm{res}_{Q=\infty^i} z_i^{-\alpha}(Q) B(\bar P, Q)$
$\sim (z_i^{-\alpha-1} + O(1))dz_i + O(1)d\bar z_i, \quad P \sim \infty^i.$

b) $\Phi_{t^{\overline{i;\alpha}}}(P) = \phi_{t^{\overline{i;\alpha}}}(P) + \psi_{t^{\overline{i;\alpha}}}(P) = \overline{\psi_{t^{i;\alpha}}(P)} + \overline{\phi_{t^{i;\alpha}}(P)} = \overline{\Phi_{t^{i;\alpha}}(P)}$.

Type 2. $i = 1, \ldots, m$.

a) $\Phi_{v^i}(P) = \phi_{v^i}(P) + \psi_{v^i}(P) = \mathrm{res}_{Q=\infty^i}\lambda(Q)\Omega(P,Q) + \mathrm{res}_{Q=\infty^i}\lambda(Q)B(\bar P, Q)$
$\sim -d\lambda + O(1)(d\bar z_i + dz_i), \quad P \sim \infty^i.$

b) $\Phi_{v^{\bar i}}(P) = \phi_{v^{\bar i}}(P) + \psi_{v^{\bar i}}(P) = \overline{\psi_{v^i}(P)} + \overline{\phi_{v^i}(P)} = \overline{\Phi_{v^i}(P)}$.

Type 3. $i = 1, \ldots, m$.

a) $\Phi_{w^i}(P) = \phi_{w^i}(P) + \psi_{w^i}(P) = \mathrm{v.p.} \displaystyle\int_{\infty^0}^{\infty^i} \Omega(P,Q) + \int_{\infty^0}^{\infty^i} B(\bar P, Q)$,
$\mathrm{res}_{\infty^i}\Phi_{w^i} = 1; \quad \mathrm{res}_{\infty^0}\Phi_{w^i} = -1.$

b) $\Phi_{w^{\bar i}}(P) = \phi_{w^{\bar i}}(P) + \psi_{w^{\bar i}}(P) = \overline{\psi_{w^i}(P)} + \overline{\phi_{w^i}(P)} = \overline{\Phi_{w^i}(P)}$.

Type 4. $k = 1, \ldots, g$.

$$\phi_{r^k}(P) + \psi_{r^k}(P) = -\oint_{a_k} \lambda(Q)\Omega(P,Q) - \oint_{a_k} \lambda(Q)B(\bar P, Q),$$

$\Phi_{r^k}(P) = \phi_{r^k}(P) + \overline{\psi_{r^k}(P)} + \psi_{r^k}(P) + \overline{\phi_{r^k}(P)}$.

Type 5. $k = 1, \ldots, g$.

$$\phi_{u^k}(P) + \psi_{u^k}(P) = \oint_{b_k} \lambda(Q)\Omega(P,Q) + \oint_{b_k} \lambda(Q)B(\bar P, Q),$$

$\Phi_{u^k}(P) = \phi_{u^k}(P) + \overline{\psi_{u^k}(P)} + \psi_{u^k}(P) + \overline{\phi_{u^k}(P)}$.

Type 6. $k = 1, \ldots, g$.

$$\Phi_{s^\alpha}(P) = \phi_{s^\alpha}(P) + \psi_{s^\alpha}(P) = \frac{1}{2\pi i}\oint_{b_\alpha} \Omega(P,Q) + \frac{1}{2\pi i}\oint_{b_\alpha} B(\bar P, Q).$$

Type 7. $k = 1, \ldots, g$.

$$\Phi_{t^\alpha}(P) = \phi_{t^\alpha}(P) + \psi_{t^\alpha}(P) = -\frac{1}{2\pi i}\oint_{a_\alpha} \Omega(P,Q) - \frac{1}{2\pi i}\oint_{a_\alpha} B(\bar P, Q).$$

For all these differentials ϕ-part is a differential with respect to a local parameter z and ψ-part is a differential with respect to $\bar z$. For brevity we denote by Φ_{ξ^A} an arbitrary differential from the above list. a- and b-periods of Φ_{ξ^A} are non-zero only for two types of primary differentials: $\oint_{a_\alpha}\Phi_{\xi^A} = \delta_{\xi^A, s^\alpha}$; $\oint_{b_\alpha}\Phi_{\xi^A} = \delta_{\xi^A, t^\alpha}$.

Flat coordinates for the metric $\langle , \rangle_{\phi+\psi}$.

$$t^{i;\alpha} = \mathrm{res}_{P=\infty^i} z_i^\alpha \left(\mathrm{v.p.} \int_{\infty^0}^{P} \phi \right) d\lambda, \qquad t^{\overline{i;\alpha}} = \tilde{\mathrm{res}}_{P=\infty^i} \bar z_i^\alpha \left(\mathrm{v.p.} \int_{\infty^0}^{P} \psi \right) d\bar\lambda,$$

$$i = 0, \dots, m, \qquad \alpha = 1, \dots, n_i,$$

$$v^i = -\text{v.p.} \int_{\infty^0}^{\infty^i} \phi, \qquad v^{\bar{i}} = -\text{v.p.} \int_{\infty^0}^{\infty^i} \psi, \qquad i = 1, \dots, m,$$

$$w^i = -\text{res}_{\infty^i} \lambda\phi, \qquad w^{\bar{i}} = -\tilde{\text{res}}_{\infty^i} \bar{\lambda}\psi, \qquad i = 1, \dots, m,$$

$$r^k = \frac{1}{2\pi i} \oint_{b_k} \phi, \quad u^k = \frac{1}{2\pi i} \oint_{a_k} \phi, \qquad k = 1, \dots, g,$$

$$s^k = -\oint_{a_k} (\lambda\phi + \bar{\lambda}\psi), \qquad t^k = -\oint_{b_k} (\lambda\phi + \bar{\lambda}\psi), \qquad k = 1, \dots, g.$$

We denote $\tilde{\text{res}} f = \overline{\text{res} \bar{f}}$ and the flat coordinates by ξ^A, i.e., $\xi^A \in \{t^{i;\alpha}, \overline{t^{i;\alpha}}; v^i, v^{\bar{i}}, w^i, w^{\bar{i}}; r^k, u^k, s^k, t^k \,|\, i = 0, \dots, m,\ \alpha = 1, \dots, n_i;\ k = 1, \dots, g\}$. The number of flat coordinates equals $2L$ by virtue of the Riemann–Hurwitz formula.

Theorem 4. *The rotation coefficients of the metric $\langle , \rangle_{\phi+\psi}$ are given by:*

$$\beta_{ij} = \frac{1}{2} \frac{\Omega(P_i, P_j)}{dx_i\, dx_j}, \qquad \beta_{i\bar{j}} = \frac{1}{2} \frac{B(P_i, \bar{P}_j)}{dx_i\, d\bar{x}_j}, \qquad \beta_{\bar{i}\bar{j}} = \overline{\beta_{ij}}.$$

They satisfy equations (2)*–*(4)*. The metrics satisfy* (1) *with the Euler field* $\sum_{k=1}^{L} (\lambda_k \partial_{\lambda_k} + \bar{\lambda}_k \partial_{\bar{\lambda}_k})$*, where in* (3) *and* (4) *the sum over the index set* $\{1, \dots, L; \bar{1}, \dots, \bar{L}\}$ *is understood.*

Prepotential of the Frobenius structure is a function $F(\xi^A)$ the third derivatives of which are given by the 3-tensor $\boldsymbol{c}$ from the definition of Frobenius manifold:

$$\frac{\partial^3 F(\xi)}{\partial \xi^A \partial \xi^B \partial \xi^C} = \boldsymbol{c}(\partial_{\xi^A}, \partial_{\xi^B}, \partial_{\xi^C}) = \langle \partial_{\xi^A} \cdot \partial_{\xi^B}, \partial_{\xi^C} \rangle_{\phi+\psi}.$$

To write an expression for prepotential we define a new pairing of multivalued differentials on $\mathcal{L}$ as follows. Let $\omega^{(i)}(P)$ be a differential on $\mathcal{L}$, such that it can be decomposed in holomorphic ($\omega^{(i)}_{1,0}$) and antiholomorphic ($\omega^{(i)}_{0,1}$) parts: $\omega^{(i)} = \omega^{(i)}_{1,0} + \omega^{(i)}_{0,1}$, which are regular outside infinities and have the following behaviour near $P \sim \infty^a$:

$$\omega^{(i)}_{1,0}(P) = \sum_{n=-n_1}^{\infty} c^{(i)}_{n,a} z_a^n dz_a + \frac{1}{n_a + 1} d\left(\sum_{n>0} r^{(i)}_{n,a} \lambda^n \log \lambda \right),$$

$$\omega^{(i)}_{0,1}(P) = \sum_{n=-n_2}^{\infty} c^{(i)}_{\bar{n},a} \bar{z}_a^n d\bar{z}_a + \frac{1}{n_a + 1} d\left(\sum_{n>0} r^{(i)}_{\bar{n},a} \bar{\lambda}^n \log \bar{\lambda} \right),$$

where $n_1, n_2 \in \mathbb{Z}$; $c^{(i)}_{n,a}$, $r^{(i)}_{n,a}$, $c^{(i)}_{\bar{n},a}$, $r^{(i)}_{\bar{n},a}$ are some coefficients. Denote also for $k = 1, \dots, g$:

$$\oint_{a_k} \omega^{(i)} = A^{(i)}_k, \qquad \oint_{b_k} \omega^{(i)} = B^{(i)}_k,$$

$$\omega^{(i)}_{1,0}(P^{a_k}) - \omega^{(i)}_{1,0}(P) = dp^{(i)}_k(\lambda), \qquad p^{(i)}_k(\lambda) = \sum_{s>0} p^{(i)}_{sk} \lambda^s,$$

$$\omega^{(i)}_{1,0}(P^{b_k}) - \omega^{(i)}_{1,0}(P) = dq^{(i)}_k(\lambda), \qquad q^{(i)}_k(\lambda) = \sum_{s>0} q^{(i)}_{sk} \lambda^s.$$

$$\omega^{(i)}_{0,1}(P^{a_k}) - \omega^{(i)}_{0,1}(P) = dp^{(i)}_{\bar{k}}(\bar{\lambda}), \qquad p^{(i)}_{\bar{k}}(\bar{\lambda}) = \sum_{s>0} p^{(i)}_{\bar{s}\bar{k}} \bar{\lambda}^s,$$

$$\omega^{(i)}_{0,1}(P^{b_k}) - \omega^{(i)}_{0,1}(P) = dq^{(i)}_{\bar{k}}(\bar{\lambda}), \qquad q^{(i)}_{\bar{k}}(\bar{\lambda}) = \sum_{s>0} q^{(i)}_{\bar{s}\bar{k}} \bar{\lambda}^s.$$

Again, $\omega(P^{a_k}) - \omega(P)$ denotes an additive transformation of differential under analytic continuation along the cycle a_k on the Riemann surface.

Definition 4. For two such differentials define the pairing:

$$\begin{aligned}\langle \omega^{(i)}\ \omega^{(j)} \rangle = \sum_{a=0}^{m} \Bigg[& \sum_{n\geq 0} \frac{c^{(i)}_{-n-2,a}}{n+1} c^{(j)}_{n,a} + c^{(i)}_{-1,a} \mathrm{v.p.} \int_{P_0}^{\infty^a} \omega_{1,0}(j) - \mathrm{v.p.} \int_{P_0}^{\infty^a} r^{(i)}_{n,a} \lambda^n \omega^{(j)}_{1,0} \\ & + \sum_{n\geq 0} \frac{c^{(i)}_{\overline{-n-2},a}}{n+1} c^{(j)}_{\bar{n},a} + c^{(i)}_{-\bar{1},a} \mathrm{v.p.} \int_{P_0}^{\infty^a} \omega^{(j)}_{0,1} - \mathrm{v.p.} \int_{P_0}^{\infty^a} r^{(i)}_{\bar{n},a} \bar{\lambda}^n \omega^{(j)}_{0,1} \Bigg] \\ & + \frac{1}{2\pi i} \sum_{k=1}^{g} \Bigg[-\oint_{a_k} q^{(i)}_k(\lambda) \omega^{(j)}_{1,0} + \oint_{a_k} q^{(i)}_{\bar{k}}(\bar{\lambda}) \omega^{(j)}_{0,1} + \oint_{b_k} p^{(i)}_k(\lambda) \omega^{(j)}_{1,0} \\ & - \oint_{b_k} p^{(i)}_{\bar{k}}(\bar{\lambda}) \omega^{(j)}_{0,1} + A^{(i)}_k \oint_{b_k} \omega^{(j)}_{1,0} - B^{(i)}_k \oint_{a_k} \omega^{(j)}_{1,0} \Bigg].\end{aligned}$$

P_0 is the point on the curve $\mathcal{L}$ such that $\lambda(P_0) = 0$; all basic cycles $\{a_k\}$ and $\{b_k\}$ start at P_0.

Consider the multivalued differential Φ on $\mathcal{L}$: $\Phi(P) = \left(\mathrm{v.p.} \int_{\infty^0}^{P} \phi\right) d\lambda + \left(\mathrm{v.p.} \int_{\infty^0}^{P} \psi\right) d\bar{\lambda}$, where ϕ and ψ are the corresponding parts of some primary differential of one of the listed types.

The pairing is defined for this differential and the following theorem holds.

Theorem 5. *The prepotential of the Frobenius structure is* $F = \frac{1}{2}\langle \Phi\ \Phi \rangle$.

Theorem 6. *For the second derivatives of the prepotential we have* $\partial_{\xi^A} \partial_{\xi^B} F = \langle \Phi_{\xi^A}\ \Phi_{\xi^B} \rangle$.

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