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One Frequency Asymptotic Solutions to Differential Equation with Deviated Argument and Slowly Varying Coefficients

Giovanni MATARAZZO †, *Anna POMPEI* ‡ *and Valeriy SAMOYLENKO* §

† *Department of Computer Engineering and Applied Mathematics, Salerno University, via Ponte Don Melillo, 84084 Fisciano (SA), Italy*
E-mail: *matarazz@diima.unisa.it*

‡ *Department of Mathematics, University of Catania, viale A. Doria 6, 95125 Catania, Italy*

§ *Department of Mechanics and Mathematics, Kyiv National Taras Shevchenko University, 64 Volodymyrs'ka Str., 01033 Kyiv-33, Ukraine*
E-mail: *vsam@univ.kiev.ua*

The object of this paper is to study a problem of constructing of an approximate one-frequency solutions for weakly nonlinear ordinary differential equations with deviated argument and slowly varying coefficients. On the basis of asymptotic techniques of nonlinear mechanics an algorithm for asymptotic integration of differential equation under consideration is given.

1 Introduction and formulation of the problem

Among numerous modern analytic methods of studying nonlinear oscillatory processes asymptotic methods are often applied [1]. The effectiveness of these methods in various fields of applied mathematics and mechanics is now recognized all over the world because those methods enable not only to construct approximate solutions of corresponding differential equations, but also to examine qualitative properties of nonlinear processes described by given equations.

In present paper we consider the problem of construction of one-frequency asymptotic approximations for weakly nonlinear multi-dimensional oscillatory systems with delay and slowly varying parameters. Such a problem is of a certain practical importance, through its connection with studying of non-stationary processes in oscillatory systems. Similar problems may occur, for example, in the systems with changing mass and inflexibility, in particular, in numerous problems of electronics and radiotechnics, connected with the problems of modulation, charged particles acceleration phenomena, control theory, etc.

The purpose of the present paper is to develop techniques for constructing of one-frequency asymptotic solution to the system of weakly nonlinear ordinary differential equations with deviated argument and slowly varying coefficients of the following form

$$\sum_{s=1}^{N}\left(\alpha_{rs}(\tau)\frac{dx_s(t)}{dt}+\beta_{rs}(\tau)\frac{dx_s(t-\sigma(\tau))}{dt}+\gamma_{rs}(\tau)x_s(t)+\delta_{rs}(\tau)x_s(t-\sigma(\tau))\right)$$
$$=\varepsilon f_r\left(\tau,\theta,x_1(t),\dots,x_N(t),x_1(t-\sigma(\tau)),\dots,x_N(t-\sigma(\tau))\right), \tag{1}$$

where ε is a small parameter; $\tau=\varepsilon t$ is *the slow time*; $d\theta/dt=\nu(\tau)$ is an instant frequency of an external periodical force; $\sigma(\tau)\geq\sigma_0>0$ is a delay; functions $f_r(\tau,\theta,x_1,\dots,x_N,y_1,\dots,y_N)$ are supposed to be 2π-periodic with respect to θ of the following representation

$$f_r(\tau,\theta,x_1,\dots,x_N,y_1,\dots,y_N)=\sum_{|m|\leq M_r}f_{rm}(\tau,x_1,\dots,x_N,y_1,\dots,y_N)e^{im\theta}.$$

We also supposed the coefficients $f_{rm}(\tau, x_1, \dots, x_N, y_1, \dots, y_N)$, $|m| \le M_r$, $r = \overline{1,N}$ to be polynomials in $x_1, \dots, x_N$; $y_1, \dots, y_N$ infinite differentiable with respect to τ. The values $\alpha_{rs} = \alpha_{rs}(\tau)$, $\beta_{rs} = \beta_{rs}(\tau)$, $\gamma_{rs} = \gamma_{rs}(\tau)$, $\delta_{rs} = \delta_{rs}(\tau)$, $r, s = \overline{1,N}$, $\nu = \nu(\tau)$, $\sigma = \sigma(\tau)$ are supposed to be sufficiently smooth in variable τ.

We assume that the characteristic equation of system (1), i.e. equation $\det \|\kappa_{rs}(\lambda)\| = 0$, where

$$\kappa_{rs}(\lambda) = \lambda\alpha_{rs}(\tau) + \gamma_{rs}(\tau) + (\lambda\beta_{rs}(\tau) + \delta_{rs}(\tau))e^{-\lambda\sigma(\tau)},$$

has $2N$ solutions $\lambda_j = \pm i\omega_j(\tau)$, $j = \overline{1,N}$, at every moment of a slow time τ. At the same time the other roots of characteristic equation are supposed to have negative real parts.

Thus, fundamental frequencies $\omega_j = \omega_j(\tau)$, $j = \overline{1,N}$, of the unperturbed system (1) (when $\varepsilon = 0$) at every τ satisfy the characteristic equation $\det \|\kappa_{rs}(i\omega_j(\tau))\| = 0$, and are smooth enough in τ. We also assume that the unperturbed system (1) has only the trivial equilibrium point $x_1 = x_2 = \dots = x_N = 0$, and the conditions of non-existence of interior resonance hold.

Let $\varphi_k^{(j)} = \varphi_k^{(j)}(\tau)$, $j, k = \overline{1,N}$ be so-called normal functions being nontrivial solutions to a system of linear algebraic homogeneous equations

$$\sum_{s=1}^{N} \kappa_{rs}(i\omega_j(\tau))\varphi_s^{(j)}(\tau) = 0, \qquad j, r = \overline{1,N},$$

and $\chi_k^{(j)} = \chi_k^{(j)}(\tau)$, $j, k = \overline{1,N}$ be solutions of the conjugate system

$$\sum_{s=1}^{s=N} \bar{\kappa}_{rs}(i\omega_j(\tau))\chi_s^{(j)}(\tau) = 0, \qquad j, r = \overline{1,N},$$

that satisfy normalization conditions.

For every value of parameter τ the unperturbed system (1) has a set of periodic solutions

$$x_s(t) = a\varphi_s^{(j)}(\tau)\cos(\omega_j(\tau)t + \varphi), \qquad j, s = \overline{1,N},$$

where $a = a(\tau)$ and $\varphi = \varphi(\tau)$ are arbitrary values called normal unperturbed oscillations of the system (1).

Basing on ideas of asymptotic techniques of nonlinear mechanics [1–4] we describe an algorithm allowing us to find an approximate solution to the problem (1) asymptotically close to normal unperturbed oscillations of system (1)

$$x_s(t) = a\varphi_s^{(1)}(\tau)\cos(\omega_1(\tau)t + \varphi), \qquad s = \overline{1,N}.$$

2 Asymptotic expansion

In the context of solving the problem of construction of approximate (asymptotic) solutions, non-resonance and resonance cases are usually studied, so solutions to the problem are found separately for each of these cases. Dependence of both of the fundamental frequency $\omega(\tau)$ and the frequency of external force $\nu(\tau)$ on slow time τ does not allow to apply such an approach to the problem under our consideration. As it is known [1,5], the problem is connected with possible transfers of the system (1) from one, for example, non-resonance state, to another resonance state, and vice versa, provoking varying of the frequencies $\omega(\tau)$, $\nu(\tau)$ with slow time τ.

Thus, taking into consideration dependence of frequencies of the system (1) on slow time τ we seek an asymptotic solution of the problem (1) in the following form:

$$x_s(t) = a\varphi_s^{(1)}(\tau)\cos\varphi + \sum_{k=1}^{\infty} \varepsilon^k U_{ks}(\tau, a, \theta, \varphi), \qquad s = \overline{1,N}, \tag{2}$$

where $\varphi = \frac{p}{q}\theta + \psi$; numbers p, q are distinct natural members and depend on correlation between frequencies $\omega_1(\tau)$, $\nu(\tau)$; functions $a(t)$ and $\psi(t)$ satisfy the following differential equations

$$\frac{da}{dt} = \sum_{k=1}^{\infty} \varepsilon^k A_k(\tau, a, \psi), \qquad \frac{d\psi}{dt} = \omega_1(\tau) - \frac{p}{q}\nu(\tau) + \sum_{k=1}^{\infty} \varepsilon^k B_k(\tau, a, \psi), \tag{3}$$

where functions $A_k(\tau, a, \psi)$, $B_k(\tau, a, \psi)$ for any $k \in \mathbb{N}$ are 2π-periodic in variable ψ.

Functions $U_{ks}(\tau, a, \theta, \varphi)$, $k \in \mathbb{N}$, $s = \overline{1, N}$, are supposed to be 2π-periodic in variables θ, ψ without first harmonics in their Fourier series expansion on φ, i.e.,

$$\int_0^{2\pi} U_{ks}(\tau, a, \theta, \varphi) e^{\pm i\varphi} \, d\varphi = 0, \qquad k \in \mathbb{N}, \quad s = \overline{1, N}. \tag{4}$$

Conditions (5) allow us to construct asymptotic solutions without secular terms.

In order to find functions $A_k(\tau, a, \psi)$, $B_k(\tau, a, \psi)$, $U_k(\tau, a, \theta, \varphi)$, $k \in \mathbb{N}$, it becomes necessary to substitute expansion (2) into equation (1), taking into account differential equations (3), expand an obtained relation into series in small parameter ε and finally equate terms at the same degree of ε. Let us introduce the following notations [6]:

$$Q_0 = \left(\omega_1(\tau) - \frac{p}{q}\nu(\tau)\right)\frac{\partial}{\partial\psi}, \qquad P_0 = \omega_1(\tau)\frac{\partial}{\partial\varphi} + \nu(\tau)\frac{\partial}{\partial\theta}, \qquad \mathcal{N}_0 = \sum_{k=0}^{\infty} \frac{(-\sigma(\tau))^{k+1}}{(k+1)!} Q_0,$$

$$M_{rs} = \alpha_{rs}(\tau)P_0 + \gamma_{rs}(\tau), \qquad N_{rs} = \beta_{rs}(\tau)P_0 + \delta_{rs}(\tau),$$

$$\begin{aligned}
\mathcal{L}_r = \sum_{s=1}^{N} \varphi_s^{(1)}(\tau) \{ &\alpha_{rs}(\tau) + \delta_{rs}(\tau)\mathcal{N}_0 \cos(\omega_1(\tau)\sigma(\tau)) \\
&+ \beta_{rs}(\tau)[(\mathcal{N}_0 Q_0 + 1)\cos(\omega_1(\tau)\sigma(\tau)) + \omega_1(\tau)\mathcal{N}_0 \sin(\omega_1(\tau)\sigma(\tau))] \}, \\
\mathcal{E}_r = \sum_{s=1}^{N} \varphi_s^{(1)}(\tau) \{ &\delta_{rs}(\tau)\mathcal{N}_0 \sin(\omega_1(\tau)\sigma(\tau)) \\
&+ \beta_{rs}(\tau)[(\mathcal{N}_0 Q_0 + 1)\sin(\omega_1(\tau)\sigma(\tau)) - \omega_1(\tau)\mathcal{N}_0 \cos(\omega_1(\tau)\sigma(\tau))] \}.
\end{aligned} \tag{5}$$

By substitution of formulas (5) into the right part of equation (1), by standard way we obtain the following relations

$$\begin{aligned}
&\sum_{s=1}^{N} [M_{rs} U_{ks}(\tau, a, \theta, \varphi) + N_{rs} U_{ks}(\tau, a, \theta - \nu\sigma, \varphi - \omega_1\sigma)] \\
&\qquad = F_{rk}(\tau, a, \theta, \varphi) - (\mathcal{L}_r A_k + a\mathcal{E}_r B_k)\cos\varphi + (\mathcal{E}_r A_k - a\mathcal{L}_r B_k)\sin\varphi,
\end{aligned} \tag{6}$$

where an explicit form of functions $F_{rk}(\tau, a, \theta, \varphi)$, $k \in \mathbb{N}$, $r = \overline{1, N}$, is found after the sequel determination of functions $A_m(\tau, a, \psi)$, $B_m(\tau, a, \psi)$, $U_m(\tau, a, \theta, \varphi)$, $m = \overline{1, k-1}$, $r = \overline{1, N}$. To realize it we have to use the 2π-periodicity property of functions $A_k(\tau, a, \psi)$, $B_k(\tau, a, \psi)$, $U_k(\tau, a, \theta, \varphi)$, $F_{rk}(\tau, a, \theta, \varphi)$, $r = \overline{1, N}$, $k \in \mathbb{N}$, with respect to variables ψ, φ, θ correspondingly, and represent these functions by means of their Fourier series as follows

$$U_{rk}(\tau, a, \theta, \varphi) = \sum_{m,n=-\infty}^{+\infty} U_{rkmn}(\tau, a) e^{i(m\theta + n\varphi)},$$

$$F_{rk}(\tau, a, \theta, \varphi) = \sum_{m,n=-\infty}^{+\infty} F_{rkmn}(\tau, a) e^{i(m\theta + n\varphi)},$$

$$A_k(\tau,a,\psi)=\sum_{n=-\infty}^{+\infty}A_{kn}(\tau,a)e^{in\psi},\qquad B_k(\tau,a,\psi)=\sum_{n=-\infty}^{+\infty}B_{kn}(\tau,a)e^{in\psi},\tag{7}$$

where

$$U_{rkmn}(\tau,a)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}U_{rk}(\tau,a,\theta,\varphi)e^{-i(m\theta+n\varphi)}\,d\theta d\varphi,$$
$$F_{rkmn}(\tau,a)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}F_{rk}(\tau,a,\theta,\varphi)e^{-i(m\theta+n\varphi)}\,d\theta d\varphi,$$
$$A_{kn}(\tau,a)=\frac{1}{2\pi}\int_0^{2\pi}A_k(\tau,a,\psi)e^{-in\psi}\,d\psi,\qquad B_{kn}(\tau,a)=\frac{1}{2\pi}\int_0^{2\pi}B_k(\tau,a,\psi)e^{-in\psi}\,d\psi.$$

Taking into consideration the relation $\varphi-\frac{p}{q}\theta=\psi$ we separate resonance and non-resonance terms in Fourier expansion (7) of functions $F_{rk}(\tau,a,\theta,\varphi)$, $k\in\mathbb{N}$, in the following way

$$F_{rk}(\tau,a,\theta,\varphi)=\sum_{mq+(n\pm1)p\neq0}F_{rkmn}(\tau,a)e^{i(m\theta+n\varphi)}+\sum_{mq+(n\pm1)p=0}F_{rkmn}(\tau,a)e^{i(m\theta+n\varphi)}.$$

In order for the equations (6) to have 2π-periodic in φ, θ solutions it is necessary that the following condition holds:

$$\sum_{r=1}^{N}\chi_r^{(1)}\left[\mathcal{L}_rA_k+a\mathcal{E}_rB_k)\cos\varphi+(\mathcal{E}_rA_k-a\mathcal{L}_rB_k)\sin\varphi\right]$$
$$=\sum_{mq+(n\pm1)p=0}F_{rkmn}(\tau,a)e^{i(m\theta+n\varphi)},\qquad m,n\in\mathbb{Z},\quad k\in\mathbb{N},\tag{8}$$

where $F_{rkmn}(\tau,a)$ are Fourier coefficients of functions $F_{rk}(\tau,a,\theta,\varphi)$.

The latter term in (8) can be given as

$$\sum_{mq+(n\pm1)p=0}F_{rkmn}(\tau,a)e^{i(m\theta+n\varphi)}$$
$$=\sum_{n=-\infty}^{+\infty}\left[F_{rk,-pn,qn-1}(\tau,a)e^{i\varphi}+F_{rk,-pn,qn+1}(\tau,a)e^{-i\varphi}\right]e^{iqn\psi}.$$

Using both condition (4) and relation (6)–(8) we find

$$\sum_{s=1}^{N}\kappa_{rs}(m\nu+n\omega_1)U_{ksmn}(\tau,a)=F_{rkmn}(\tau,a),\qquad r=\overline{1,N},\tag{9}$$

where $mq+(n\pm1)p\neq0$, $m,n\in\mathbb{Z}$.

Evidently, the system of algebraic equations (9) has a unique solution

$$U_{ksmn}(\tau,a)=\sum_{r=1}D_{ks}(m\nu+n\omega_1)F_{rkmn}(\tau,a)\left(\det\|\kappa_{rs}(m\nu+n\omega_1)\|\right)^{-1},$$

where $D_{ks}(\lambda)$ are corresponding minors of the determinant $\det\|\kappa_{rs}(\lambda)\|$, if $mq+(n\pm1)p\neq0$, $m,n\in\mathbb{Z}$. In case $mq+(n\pm1)p=0$, $m,n\in\mathbb{Z}$, we put $U_{ksmn}(\tau,a)=0$.

Functions $A_k(\tau,a,\psi)$, $B_k(\tau,a,\psi)$, $k\in\mathbb{N}$, are defined from the system of differential equations

$$\mathcal{L}A_k+a\mathcal{E}B_k=G_k(\tau,a,\psi),\qquad \mathcal{E}A_k-a\mathcal{L}B_k=H_k(\tau,a,\psi),\tag{10}$$

where

$$\mathcal{L} = \sum_{r=1}^{N} \chi_r^{(1)}(\tau)\mathcal{L}_r, \qquad \mathcal{E} = \sum_{r=1}^{N} \chi_r^{(1)}(\tau)\mathcal{E}_r,$$

$$G_k(\tau, a, \psi) = \sum_{r=1}^{N} \frac{\chi_r^{(1)}(\tau)}{4\pi^2} \sum_{m=-\infty}^{+\infty} \int_0^{2\pi}\int_0^{2\pi} F_{rk}(\tau, a, \theta, \varphi) e^{im(p\theta - q\varphi)}\,(e^{i\varphi} + e^{-i\varphi}) e^{imq\psi}\; d\theta\, d\varphi\,,$$

$$H_k(\tau, a, \psi) = \sum_{r=1}^{N} \frac{i\chi_r^{(1)}(\tau)}{4\pi^2} \sum_{m=-\infty}^{+\infty} \int_0^{2\pi}\int_0^{2\pi} F_{rk}(\tau, a, \theta, \varphi) e^{im(p\theta - q\varphi)}\,(e^{i\varphi} - e^{-i\varphi}) e^{imq\psi}\; d\theta\, d\varphi,$$

To solve equations (10), we use the Fourier representations (7) and easily find

$$A_k(\tau, a, \psi) = \sum_{n=-\infty}^{+\infty} \frac{1}{4\pi^3} e^{in\psi}\left(\mathcal{L}_{0n}^2 + \mathcal{E}_{0n}^2\right)^{-1} \int_0^{2\pi} \sum_{m=-\infty}^{+\infty} \int_0^{2\pi}\int_0^{2\pi} \sum_{r=1}^{N} \chi_r^{(1)}(\tau)$$
$$\times F_{rk}(\tau, a, \theta, \varphi) e^{im(p\theta - q\varphi)} (\mathcal{L}_{0n}\cos\varphi - \mathcal{E}_{0n}\sin\varphi) e^{i(mq-n)\psi}\; d\theta d\varphi d\psi,$$

$$B_k(\tau, a, \psi) = \sum_{n=-\infty}^{+\infty} \frac{1}{4a\pi^3} e^{in\psi}\left(\mathcal{L}_{0n}^2 + \mathcal{E}_{0n}^2\right)^{-1} \int_0^{2\pi} \sum_{m=-\infty}^{+\infty} \int_0^{2\pi}\int_0^{2\pi} \sum_{r=1}^{N} \chi_r^{(1)}(\tau)$$
$$\times F_{rk}(\tau, a, \theta, \varphi) e^{im(p\theta - q\varphi)} (\mathcal{L}_{0n}\sin\varphi + \mathcal{E}_{0n}\cos\varphi) e^{i(mq-n)\psi}\; d\theta d\varphi d\psi,$$

where

$$\mathcal{L}_{0n} = \sum_{s,r=1}^{N} \chi_r^{(1)}(\tau)\varphi_s^{(1)}(\tau)\Big[\alpha_{rs}(\tau) + \delta_{rs}(\tau)\rho(nl(\tau))\cos(\omega_1(\tau)\sigma(\tau)) +$$
$$+ \beta_{rs}(\tau)e^{-n\sigma(\tau)l(\tau)}\cos(\omega_1(\tau)\sigma(\tau)) + \beta_{rs}(\tau)\omega_1(\tau)\rho(nl(\tau))\sin(\omega_1(\tau)\sigma(\tau))\Big],$$

$$\mathcal{E}_{0n} = \sum_{s,r=1}^{N} \chi_r^{(1)}(\tau)\varphi_s^{(1)}(\tau)\Big[\delta_{rs}(\tau)\rho(nl(\tau))\sin(\omega_1(\tau)\sigma(\tau))$$
$$+ \beta_{rs}(\tau)e^{-n\sigma(\tau)l(\tau)}\sin(\omega_1(\tau)\sigma(\tau)) - \beta_{rs}(\tau)\omega_1(\tau)\rho(nl(\tau))\cos(\omega_1(\tau)\sigma(\tau))\Big],$$

$$\rho(nl) = \begin{cases} (e^{-n\sigma l(\tau)} - 1)/(inl) & \text{if} \quad nl \neq 0, \\ 0 & \text{if} \quad nl = 0. \end{cases}$$

Thus, the problem of construction of one-frequency solutions to equations (1) asymptotically close to normal unperturbed oscillations of system (1) is solved.

3 Conclusion

In the paper a problem of construction of an approximate one-frequency solutions to weakly nonlinear ordinary differential equations with deviated argument and slowly varying coefficients is studied. An algorithm for asymptotic integration of differential equation under consideration is given.

The approach proposed above allows to build asymptotical solution to weakly nonlinear ordinary the first order differential equations with deviated argument and slowly varying coefficients. Having defining approximate solution (functions $A_k(\tau, a, \psi)$, $B_k(\tau, a, \psi)$, $U_{rk}(\tau, a, \theta, \varphi)$, $k \in \mathbb{N}$, $r = \overline{1, N}$) it is possible to study stationary regimes and their stability as well as processes connected with passing through resonance zones [2].

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