Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 3, 1429–1434

The WKB Method for the Dirac Equation with Vector-Scalar Potentials in 2+1 and 3+1 Dimensions

Olexander K. REITY, Vasyl' V. RUBISH and Svitlana I. MYHALYNA

Uzhgorod National University, 32 Voloshyna Str., Uzhgorod, Ukraine
E-mail: *reiti@univ.uzhgorod.ua, vrubish@univ.uzhgorod.ua*

In the framework of the Dirac approach we have developed the relativistic version of the WKB method for centrally symmetrical potential with the mixed Lorentz structure. We have obtained relativistic wavefunctions of light quark and the new quantization rule containing the spin-orbit interaction. These gives us the possibility of finding the energy levels and decay width of hydrogen-like quark-systems.

1 Introduction

Dirac equations with potentials of a concrete type, for which it is possible to write the exact solution, are encountered quite seldom. Most often either numerical or asymptotic methods are used for searching the solutions. In many theoretical and applied problems the possibility of obtaining the asymptotic solution allows to carry out the most complete analysis of a problem. Therefore, hardly there is a necessity to explain in detail importance of creating and investigating asymptotic methods for the solving the Dirac equation. The Wentzel–Kramers–Brillouin quasi-classical approximation (or WKB method) is one of basic and most universal asymptotic methods of solving problems of theoretical and mathematical physics (see, for example, [1–4]), for which the exact solutions are either unknown or rather difficult to find. As it is known [1–4], in the case of a Coulomb field this method has high accuracy even for small values of quantum numbers. In contrast to the perturbation theory, the given approach is not connected with a smallness of interaction and consequently has wider applicability area allowing to study qualitative legitimacies in behaviour and of properties of quantum mechanical systems. The new application of quasi-classical approach can be low-energy sector of QCD (the energy spectrum of hadrons, confinement study, decay widths of hadrons), where standard approaches based on perturbative theories are inapplicable because of the fact that interaction between quarks in this area is not small.

2 The WKB method for the Dirac equation in a centrally symmetrical field

We obtain the formulae of the quasi-classical approximation for solutions of the Dirac equation with potential of centrally symmetry having mixed Lorenz structure. For this purpose we search wavefunctions of the stationary states (in standard representation) in the form of

$$\Phi = r^{-1} \begin{pmatrix} F(r)\Omega_{jlM}(\boldsymbol{n}) \\ iG(r)\Omega_{jl'M}(\boldsymbol{n}) \end{pmatrix}, \tag{1}$$

where Ω is the spherical spinor, j and M are the total angular moment and projection of j ($j = l \pm 1/2$) respectively, l is the orbital moment ($l + l' = 2j$), $\boldsymbol{n} = \boldsymbol{r}/r$.

After the separation of the angular variables the Dirac equation with centrally symmetrical potential containing both vector $V(r)$ and scalar $S(r)$ parts takes the form ($\hbar = c = 1$)

$$\frac{dF}{dr} + \frac{k}{r}F - \frac{1}{\hbar}\left[(E - V(r)) + (m + S(r))\right]G = 0,$$
$$\frac{dG}{dr} - \frac{k}{r}G + \frac{1}{\hbar}\left[(E - V(r)) - (m + S(r))\right]F = 0, \tag{2}$$

where F and G are the radial functions, $k = \mp(j + 1/2)$ for states with $j = l \pm 1/2$, E is the energy of level.

For finding quasi-classical solutions of the system (2) it is convenient to write equations (2) in the matrix form [5]:

$$\Psi' = \frac{1}{\hbar}D\Psi, \qquad \Psi = \left\{ \begin{array}{c} F \\ G \end{array} \right\},$$
$$D = \left(\begin{array}{cc} -\hbar k/r & E - V(r) + m + S(r) \\ -E + V(r) + m + S(r) & \hbar k/r \end{array} \right). \tag{3}$$

Here a Planck constant $\hbar$ remains in the explicit form, the prime denotes the derivative with respect to r. We will regard the solution of the matrix equation (3) as the formal expansion in powers of $\hbar$:

$$\Psi = \varphi \exp\left(\int^r y dr\right), \qquad y(r) = \frac{1}{\hbar}y_{-1}(r) + y_0(r) + \hbar y_1(r) + \cdots,$$
$$\varphi(r) = \sum_{n=0}^{\infty} \hbar^n \varphi^{(n)}(r), \tag{4}$$

where the upper (lower) component $\varphi_F^{(n)}$ ($\varphi_G^{(n)}$) of the vector $\varphi^{(n)}$ corresponds to the radial wavefunction $F(G)$. Having substituted (4) into (3) and equated to zero the coefficients of each power of $\hbar$, we arrive at the recurrent system

$$(D - y_{-1})\varphi^{(0)} = 0, \qquad (D - y_{-1})\varphi^{(n+1)} = \varphi^{(n)\prime} + \sum_{k=0}^{n} y_{n-k}\varphi^{(k)}, \quad n = 0, 1, \ldots. \tag{5}$$

Using the first two equation of system (5) by means of the left and right vectors technique we find the terms y_{-1}, y_0 and $\varphi^{(0)}$. So we have obtained (up to a normalization constant)

$$\Psi = \left\{ \begin{array}{c} F \\ G \end{array} \right\} = \left[2q\left(q \pm \frac{k}{r}\right)\right]^{-1/2}$$
$$\times \exp\left\{\pm \int^r q dr + \frac{1}{2}\int^r \frac{(m+S)V' + (E-V)S'}{q\left(q \pm kr^{-1}\right)} dr\right\} \left(\begin{array}{c} m + S + E - V \\ kr^{-1} \pm q \end{array} \right), \tag{6}$$
$$q = \sqrt{(m+S)^2 - (E-V)^2 + (k/r)^2}, \tag{7}$$

(here and hereinafter $\hbar = 1$).

The effective energy and effective potential

$$E_{\text{ef}} = \frac{E^2 - m^2}{2m}, \qquad U_{\text{ef}}(r, E) = \frac{E}{m}V + S + \frac{S^2 - V^2}{2m} + \frac{k^2}{2mr^2} \tag{8}$$

correspond to the Dirac system (2). So $U(r, E)$ looks like a potential with a barrier (Fig. 1). We consider the most general case, when the effective potential is barrier type (Fig. 1). Then wavefunction has the different form in the three regions: 1) potential well $r_0 < r < r_-$ ($q^2 < 0$); 2) the below-barrier region $r_- < r < r_+$ ($q^2 > 0$); 3) the classically allowed region with continuum spectrum $r > r_+$ ($q^2 < 0$), where r_0, r_- and r_+ are turning points.

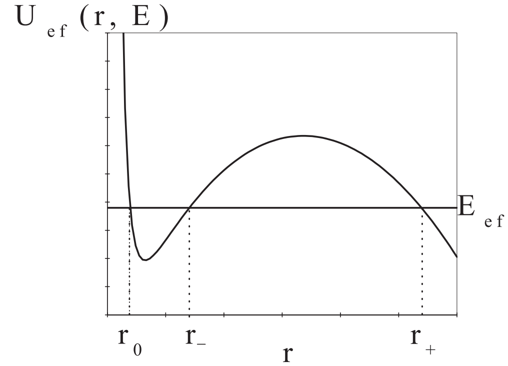

Figure 1. The type of the effective potential $U_{\mathrm{ef}}(r, E)$.

3 The wavefunction of the Dirac particle in classically allowed and forbidden regions

The wavefunction of state has the different form in the various regions.

I. The region $r_0 < r < r_-$ is classically allowed; there the wavefunctions (6) oscillate

$$F = C_1^{\pm} \left[\frac{E - V + m + S}{p} \right]^{1/2} \cos \Theta_1, \qquad G = C_1^{\pm} \operatorname{sgn} k \left[\frac{E - V - m - S}{p} \right]^{1/2} \cos \Theta_2, \quad (9)$$

where

$$p(r) = \sqrt{(E - V)^2 - (m + S)^2 - (k/r)^2} \qquad (10)$$

is the quasi-classical momentum for the radial motion of a particle, $C_1^{\pm}$ is a normalization constant,

$$\Theta_1 = \int_{r_-}^{r} \left(p + \frac{kw}{pr} \right) dr + \frac{\pi}{4}, \qquad \Theta_2 = \int_{r_-}^{r} \left(p + \frac{k\widetilde{w}}{pr} \right) dr + \frac{\pi}{4}, \qquad (11)$$

$$w = \frac{1}{2} \left(\frac{V' - S'}{m + S + E - V} - \frac{1}{r} \right), \qquad \widetilde{w} = \frac{1}{2} \left(\frac{V' + S'}{m + S - E + V} + \frac{1}{r} \right).$$

Signs $\pm$ correspond to values $k > 0$ and $k < 0$ respectively. If the level width Γ is small (it will be shown later) the wavefunction of quasi-stationary state can be normalized on a single particle localized in the region I, neglecting its penetrability into the classically forbidden regions $r < r_0$ and $r > r_-$ [1]

$$\int_{r_0}^{r_-} \left(F^2 + G^2 \right) dr = 1. \qquad (12)$$

Here $\cos^2 \Theta_i (r)$ can be replaced with average value 1/2:

$$\left| C_1^{\pm} \right| = \left[\int_{r_0}^{r_-} \frac{E - V(r)}{p(r)} dr \right]^{-1/2} = \left(\frac{2}{T} \right)^{1/2}, \qquad (13)$$

where T is the oscillation period of a relativistic particle inside a potential well.

II. The below-barrier region $r_- < r < r_+$ is classically forbidden. Here $p = iq$, and quantities q, y_{-1} and y_0 are real. As it is known [1], the wavefunction should exponentially decrease inside this region. So the solutions of the Dirac system (2) in the below-barrier region for $k < 0$ are

$$\Psi = \frac{C_2^-}{\sqrt{qQ}} \exp \left[- \int_{r_+}^{r} \left(q - \frac{(m + S) V' + (E - V) S'}{2qQ} \right) dr \right] \begin{pmatrix} m + S + E - V \\ -Q \end{pmatrix}, \qquad (14)$$

where $Q = q + |k|/r$.

III. The result for region of continuum ($r > r_+$) is the most interesting. Here the wavefunction corresponds to the divergent wave (taking off particle): (for $k < 0$)

$$\Psi = \frac{C_3^-}{\sqrt{pP}} \exp\left[\int_{r_+}^{r} \left(ip - \frac{(m+S)\,V' + (E-V)\,S'}{2pP}\right) dr\right] \begin{pmatrix} m+S+E-V \\ iP \end{pmatrix}, \tag{15}$$

where $P = p + i\,|k|\,/r$.

For bypass of these points and matching the solutions we employ the usual method [2]. Closely to the r_- and r_+ the system (2) is reduced to the Schrödinger equation with the effective potential linearly depending on $r - r_\pm$, the solution of which is expressed by the Airy function; one can use a the more elegant Zwaan method [1, 3]. So the relation between the constants in various regions is of the form

$$\begin{aligned} C_2^\pm = -iC_3^\pm = \mp\frac{C_1^\pm}{2} &\left[\frac{E - V(r_-) + m + S(r_-)}{|k|\,r_-^{-1}}\right]^{\pm\frac{1}{2}} \\ &\times \exp\left[-\int_{r_-}^{r_+} \left(q \pm \frac{(m+S)\,V' + (E-V)\,S'}{2qQ}\right) dr\right]. \end{aligned} \tag{16}$$

Let us find the equation determining the real part of the level energy E and width Γ of quasi-stationary levels $E = E_r - i\Gamma/2$. Neglecting the barrier penetrability from (9) we obtain the quantization condition:

$$\int_{r_0}^{r_-} \left(p + \frac{kw}{pr}\right) dr = \left(n_r + \frac{1}{2}\right)\pi, \qquad n_r = 0, 1, 2, \ldots, \tag{17}$$

where n_r is the radial quantum number.

Let us proceed to calculation of the level width Γ. Having used the explicit expressions (15) for functions F and G, relation between normalization constants (16) and expression for $C_1^\pm$ (13), we get the tunneling probability:

$$\Gamma = -2\mathrm{Im}\,[G^* F]_{r\to\infty} = \frac{1}{T}\exp\left[-2\int_{r_-}^{r_+} \left(q - \frac{kw}{qr}\right) dr\right]. \tag{18}$$

However, the formulae (9)–(18) essentially differ from the formulae of nonrelativistic quasi-classics (in particular, by the relativistic expression for the quasimomentum p, taking into account spin-orbit interaction and additional pre-exponent multiplier) and are more complicated. Their application to concrete problems does not encounter difficulties, since all quantities in functions F and G are expressed in quadratures.

In (2+1) dimensions the Dirac system differs from (2) by replacements $G \to -G$ and $k \to -(l + 1/2)$, where $l + 1/2$ is the eigenvalue of the total angular moment J_z [6]. So all the expressions for wavefunctions, quantization condition and decay width in (2+1)-dimensional case can be obtained from formulae (9)–(18) by these replacements.

4 Approbation of the results obtained

4.1. For testing our version of the WKB method, let us obtain the Sommerfeld–Dirac formula known in atomic physics. We choose $V(r) = -\frac{\alpha Z}{r}$ and $S(r) = 0$ ($\alpha \approx 1/137$ is the fine structure constant). Having calculated the integral from r_0 to r_- in the quantization condition (17), we arrive at the Sommerfeld–Dirac formula

$$E = m\left[1 + \left(\frac{\alpha Z}{n + \sqrt{k^2 - (\alpha Z)^2}}\right)^2\right]^{-1/2},$$

$$n = n_r + \frac{(1 + \operatorname{sgn}\, k)}{2} = \begin{cases} 0, 1, 2\ldots, & \text{for } k < 0, \\ 1, 2\ldots, & \text{for } k > 0. \end{cases}$$

Here $n_r = 0, 1, 2, \ldots$ is the radial quantum number from the quantization condition (17).

4.2. There is one more interesting case, in which it is possible to get an exact solution of the Dirac equation with potential of the type $V(r) = -\frac{\alpha}{r}$, $S(r) = -\frac{\alpha'}{r}$. In the same way as that by means of which the Coulomb potential $V(r)$ is derived from the exchange of massless photon between the nucleus and the leptons orbiting around it, the scalar potential $S(r)$ is created by the exchange of massless scalar mesons. The quantization condition (17) gives result coinciding with the expression obtained in [7]:

$$E = m\left\{\frac{-\alpha\alpha'}{\alpha^2 + (n+\gamma)^2} \mp \left[\left(\frac{\alpha\alpha'}{\alpha^2 + (n+\gamma)^2}\right)^2 - \frac{\alpha'^2 - (n+\gamma)^2}{\alpha^2 + (n+\gamma)^2}\right]^{1/2}\right\},$$

$$\gamma = \sqrt{k^2 - \alpha^2 + \alpha'^2}.$$

4.3. We consider the potential $V(r) = S(r) = ar^2/4$, a is the force of an oscillator. Integration in quantization condition (17) from r_0 to r_- gives the following result

$$\frac{(E-m)}{2}\sqrt{\frac{E+m}{2a}} - \frac{|k|}{2} - \frac{1}{4}\operatorname{sgn} k = n_r + \frac{1}{2},$$

input $\mathcal{K} = |k| + (1 + \operatorname{sgn} k)/2$, get

$$(E-m)\sqrt{2(E+m)} = (4n_r + 2\mathcal{K} + 1)\sqrt{a}.$$

The latter equation for energy is cubic, its real solution is

$$E_{n_r,\mathcal{K}} = \left(2m + 8\cdot 2^{2/3} m^2 A^{-1/3} + 2^{1/3} A^{1/3}\right)/6,$$

where $A = -B + \sqrt{B^2 - 1024m^6}$, $B = 32m^3 - 27a(1 + 2\mathcal{K} + 4n_r)^2$. This expression completely coincides with result obtained in [8].

4.4. Let us consider a motion of a massless fermion in the scalar field $S(r) = -\frac{\alpha'}{r} + \sigma r$, $\sigma > 0$; $V(r) = 0$. Integration in quantization condition (17) from r_0 to r_- gives following result

$$\frac{E^2 + 2\sigma(\alpha' - \gamma)}{4\sigma} - \frac{k}{\sigma(r_0 + r_-)\pi}\left[2r_0\left(\frac{\Pi(\nu_+^2, \chi)}{r_0^2 - P_+^2} + \frac{\Pi(\nu_-^2, \chi)}{r_0^2 - P_-^2}\right) - \left(\frac{1}{r_0 + P_+} + \frac{1}{r_0 + P_-}\right)F(\chi)\right] = \left(n_r + \frac{1}{2}\right), \tag{19}$$

where

$$r_{0,-} = \frac{1}{2^{1/2}\sigma}\sqrt{E^2 + 2\alpha'\sigma \mp \sqrt{(E^2 + 2\alpha'\sigma)^2 - (2\sigma\gamma)^2}}, \qquad \gamma = \sqrt{k^2 + \alpha'^2},$$

$$P_\pm = \frac{1}{2\sigma}(-E \pm \sqrt{E^2 + 4\alpha'\sigma}), \quad \chi = \sqrt{\frac{E^2 + 2\sigma(\alpha' - \gamma)}{E^2 + 2\sigma(\alpha' + \gamma)}}, \qquad \nu_\pm^2 = \chi\,\frac{P_\pm + r_0}{P_\pm - r_0}.$$

Calculating the integrals approximately in (17), one obtains the following spectrum instead of (19)

$$\frac{E_{n_r,k}^2}{4\sigma} = n_r + \frac{1}{2} + \frac{\gamma - \alpha'}{2} + \frac{k}{4\gamma} + \frac{\sigma k}{2\pi E_{n_r,k}^2}\left(0.38 + \ln\frac{E_{n_r,k}^2}{\sigma\gamma}\right) + O\left(\left(\frac{\sigma\gamma}{E_{n_r,k}^2}\right)^2\right). \tag{20}$$

The form (20) is exact for small $(\gamma\sigma/E_{n_r,k}^2)^2$, and the accuracy can be tested by comparison with the exact formula (19); two sets of lowest levels with $k = \pm 1$ coincide within 2 % and for higher levels the precision is smaller than 1 %. At $\alpha' = 0$ expression (20) is reduced to Simonov's result [9].

In case of $E_{n_r,k}^2 \gg \sigma\gamma$ in formula (20) expression $\left(0.38 + \ln E_{n_r,k}^2/\sigma\gamma\right)/\pi$ tends to 1 and one can obtain following expression for energies

$$\varepsilon_{n_r,k} = \frac{E_{n_r,k}}{\sqrt{\sigma}} = \pm\sqrt{N - \alpha' + [(N-\alpha')^2 + 2k]^{1/2}}, \qquad N = 2n_r + 1 + \gamma + \frac{k}{2\gamma}. \tag{21}$$

Acknowledgements

This work was partially supported by INTAS grant No. 99-01326.

[1] Landau L.D. and Lifshitz E.M., Course of theoretical physics, Vol. 3, Quantum mechanics. Non-relativistic theory, London-Paris, Pergamon Press, 1958.

[2] Migdal A.B., Qualitative methods in quantum theory, Moscow, Nauka, 1975.

[3] Heading J., An introduction to phase-integral methods, London, Methuen, 1962.

[4] Fröman N. and Fröman O., JWKB approximation, Amsterdam, North-Holland Publishing Company, 1965.

[5] Mur V. and Popov V., Semiclassical approximation for the Dirac equation in strong fields, *Yad. Fiz.*, 1978, V.28, 837–849.

[6] Reity O.K. and Lazur V.Yu., WKB method for the Dirac equation with the cental-symmetrical potential and its application to the theoty of two dimensional supercritical atoms, in Proceedings of Fourth International Conference "Symmetry in Nonlinear Mathematical Physics" (9–15 July, 2001, Kyiv), Editors A.G. Nikitin, V.M. Boyko and R.O. Popovych, Kyiv, Institute of Mathematics, 2002, V.43, Part 2, 676–682.

[7] Greiner W., Muller B. and Rafelski J., Quantum electrodynamics of strong fields, Berlin, Springer, 1985.

[8] Qiang W.-C., Bound states of the Klein–Gordon and Dirac equations for scalar and vector harmonic oscillator potentials, *Chinese Physics*, 2002, V.11, N 8, 757–759.

[9] Simonov Yu.A., Theory of light quarks in the confining vacuum, *Yad. Fiz.*, 1997, V.60, N 12, 2252–2276.