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Modified Hypergeometric Equations Arising from the Markoff Theory

Serge PERRINE

FTRD, 38-40 rue du General Leclerc, 92794 Issy les Moulineaux Cedex 9, France
E-mail: *serge.perrine@francetelecom.com*

After recalling what the Markoff theory is, this article summarizes some links which exist with the group $GL(2,\mathbb{Z})$ of 2×2 matrices with integer coefficients and determinant ± 1 and with its subgroups $SL(2,\mathbb{Z})$ and the triangle group $\mathbf{T}_3$. Then we consider the relation with conformal punctured tori. The main part of the article is about the monodromy representation of the Poincaré group of such a torus. We give the corresponding solution of the associated Riemann–Hilbert problem and the corresponding differential operator whose spectral analysis remains to be done. We conclude quoting the Hilbert's 22^{nd} problem and some information about the accessory parameter problem.

1 Introduction

For a real quadratic form $f(x,y) = ax^2 + bxy + cy^2 \in \mathbb{R}[x,y]$, the issue of knowing the minimal value of $|f(x,y)|$ when x and y are non-zero integers is classical. When $f(x,y)$ is a definite form, i.e. $\Delta(f) = b^2 - 4ac < 0$, the problem was solved by J.L. Lagrange and C. Hermite [18]:

$$C(f) = \frac{\inf_{(x,y)\in\mathbb{Z}^2/\{(0,0)\}} |f(x,y)|}{\sqrt{|\Delta(f)|}} \le \frac{1}{\sqrt{3}} = C(x^2+xy+y^2).$$

It has been shown [3, p. 33] that for any $\rho \in]0,(1/\sqrt{3})]$, we can find $f(x,y)\in\mathbb{R}[x,y]$ a quadratic form verifying $\rho = C(f)$. When $f(x,y)$ is a indefinite form, i.e. $\Delta(f) = b^2 - 4ac > 0$, A. Korkine and G. Zolotareff [23] demonstrated:

$$C(f) \le \frac{1}{\sqrt{5}} = C(x^2 - xy - y^2) = C(f_0),$$

an isolated value giving also $C(f) \le 1/\sqrt{8}$ for any other form f not $GL(2,\mathbb{Z})$-equivalent to f_0. Trying to understand this phenomenon motivated A.A. Markoff to write [27]. He described an infinity of values $C(f_i)_{i\in\mathbb{N}}$ comprised between $(1/\sqrt{5})$ and $(1/3)$ and having the same properties as $C(f_0)$. These values are isolated and convergent towards $(1/3)$. They can be built thanks to the tree of solutions of the Diophantine equation, so-called Markoff equation [9]:

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2.$$

For values $C(f)$ less than $(1/3)$, the author has shown that more general diophantine equations give an insight, sometimes with theories similar to the Markoff one [32], but with some complication. Moreover, a geometrical interpretation of such results has been found, similar to what was done by H. Cohn [8] for the classical Markoff theory. The general situation can be understood by the Teichmüller theory on the topological punctured torus $\mathcal{T}^\bullet$ (see for example [20]). This topological object is quite frequent in physical problems, for example when linked to the K.A.M. theorem [1], and some work has been done after the observation that the Markoff theory could be useful in order to understand the behavior of some oscillators [33]. It was possible to realize that two types of geometric punctured tori exist, we called them hyperbolic and parabolic. The Markoff theory is then linked to the parabolic case, and geometrically to special fuchsian groups, the Fricke groups Γ as defined in [37]:

(1): Γ is isomorphic to a free group with two generators $\mathbf{F}_2 = \mathbb{Z} * \mathbb{Z}$.

(2): The Riemann surface $\mathcal{H}/\Gamma$ (where $\mathcal{H}$ is the Poincaré half-plane) is homeomorphic to a punctured torus.

The closed geodesics on such Riemann surfaces are linked to indefinite quadratic forms f and to the associated Markoff constant $C(f)$, which can be seen as shorter length of such geodesics [42]. The Markoff theory gives an explanation [38] to the quantification which appears when changing from such a geodesic to another: no continuous deformation is possible on the torus because of the puncture. Such remarks are the basis of the interest of physicists in this subject [17]. In another direction, it is known that the Markoff theory has links with the study of exceptional bundles and helices of the projective plane $P_2(\mathbb{C})$ (see [36, 30, 31, 14, 15, 11]), and also with the spectrum of Hermitian operators [22], and this is also important for physics [39]. A project that we had a long time ago was to build a common interpretation of such remarks in order to get the set of all Markoff constants of indefinite quadratic forms, the Markoff spectrum, as the spectrum of some operator on a Hilbert space. The reason for it is that the Markoff spectrum seems like the spectrum of some operators. It has a discrete part from $(1/\sqrt{5})$ to $(1/3)$, then a Cantorian part from $(1/3)$ to the Freiman number β, which is:

$$\beta^{-1} = 4 + \frac{253589820 + 283748\sqrt{462}}{491993569}.$$

From β to 0 the spectrum is continuous, any real number is a Markoff constant [9]. The present article gives hints of the possibility to implement such a project, building a possible operator to consider.

Looking at the link with the Fuchsian groups that we mentioned, the Markoff theory can be described [32] thanks to the two following matrices generating in $SL(2,\mathbb{Z})$ a free group isomorphic to $\mathbf{F}_2$ which is $[SL(2,\mathbb{Z}), SL(2,\mathbb{Z})]$:

$$A_0 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

These matrices give rise to a representation $\rho : \mathrm{Aut}(\mathbf{F}_2) \longrightarrow GL(2,\mathbb{Z})$, and we have described the algebraic importance of this situation in [32]. In this article, the main goal is to give a differential equation whose former representation ρ is the monodromy representation. With such a construction, we hope to understand the Lamé equations appearing for the accessory parameters of punctured tori [21, 34]. Such equations have similarities with hypergeometric equations and also with some Schrödinger equations whose monodromy group has recently been studied [44]. Also we could soon build an Hamiltonian interpretation in the spirit of L.D. Fadeev [12] and others enabling us to realize our above mentionned project.

2 Considering the triangle group

The Markoff equation gives a complete tree of integer solutions thanks to the solution $(1, 1, 1)$ and to the three transformations which are involutions:

$$\begin{aligned} X &: (m, m_1, m_2) \longmapsto ((3m_1m_2 - m, m_1, m_2), \\ Y &: (m, m_1, m_2) \longmapsto (m, 3mm_2 - m_1, m_2), \\ Z &: (m, m_1, m_2) \longmapsto (m, m_1, 3mm_1 - m_2). \end{aligned}$$

The involutions X, Y et Z, give rise to the triangle group $\mathbf{T}_3 = \mathbf{C}_2 * \mathbf{C}_2 * \mathbf{C}_2$, which is the free product of three cyclic groups $\mathbf{C}_2$ with two elements. In [32], we showed how this group $\mathbf{T}_3$ is linked to the group of 2×2 matrices $GL(2, \mathbb{Z})$. For this we used an abelianisation morphism π' from the automorphism group $\mathrm{Aut}(\mathbf{F}_2)$ to $GL(2, \mathbb{Z})$, and two matrices generating the dihedral group $\mathbf{D}_6$ with 12 elements inside $GL(2, \mathbb{Z})$:

$$\pi'(t) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \pi'(o) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

We also defined:

$$\pi'(X) = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}, \qquad \pi'(Y) = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \qquad \pi'(Z) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The group $\mathbf{T}_3$ acts in $GL(2, \mathbb{Z})$ defining with $ch = ch(X, Y, Z) \in \mathbf{T}_3$:

$$ch(\pi'(X), \pi'(Y), \pi'(Z)) = \pi'(ch(X, Y, Z)) \in \pi'(\mathbf{T}_3).$$

It gives a ternary decomposition in $GL(2, \mathbb{Z})$ using the triangle group (see [32] for demonstration):

Proposition 1. *Every element $V \in GL(2, \mathbb{Z})$ has a unique decomposition*

$$\pi'(o)^h \pi'(t)^k ch(\pi'(X), \pi'(Y), \pi'(Z)), \qquad \text{where} \quad h = 0, 1; \quad k = 0, 1, \ldots, 5; \quad ch \in \mathbf{T}_3.$$

The elements of $\pi'(\mathbf{T}_3)$ are characterized by the conditions $h = 0$ et $k = 0$. The group $\pi'(\mathbf{T}_3)$ is not normal inside $GL(2, \mathbb{Z})$. It is isomorphic by π' to the group $\mathbf{T}_3$. The elements of the group $\mathbf{D}_6$, not normal in $GL(2, \mathbb{Z})$, are characterized by the condition $ch(\pi'(X), \pi'(Y), \pi'(Z)) = \mathbf{1}_2$.

The group $\mathbf{D}_6$ introduces two equivalence relations in $GL(2, \mathbb{Z})$, which are defined with $V_1 \ \Re_{\mathbf{D}_6} \ V_2 \quad \Leftrightarrow \quad V_1 V_2^{-1} \in \mathbf{D}_6$ and $V_1 \ {}_{\mathbf{D}_6}\Re \ V_2 \quad \Leftrightarrow \quad V_1^{-1} V_2 \in \mathbf{D}_6$. The quotients $GL(2, \mathbb{Z})/\Re_{\mathbf{D}_6}$ and $GL(2, \mathbb{Z})/{}_{\mathbf{D}_6}\Re$ are equipotent, but different because $\mathbf{D}_6$ is not a normal subgroup of $GL(2, \mathbb{Z})$. Each $V \in GL(2, \mathbb{Z})$ defines a unique $ch(\pi'(X), \pi'(Y), \pi'(Z)) \in \pi'(\mathbf{T}_3)$, such that $V \ \Re_{\mathbf{D}_6} \ ch(\pi'(X), \pi'(Y), \pi'(Z))$. Hence, we get a description of the complete tree of the Markoff theory:

Proposition 2. *The group $\mathbf{T}_3$ is equipotent to the quotient (right or left) of the group $GL(2, \mathbb{Z})$ by its non-normal subgroup $\mathbf{D}_6$. It is an homogeneous $GL(2, \mathbb{Z})$-space. But also it can be considered a subgroup of $GL(2, \mathbb{Z})$, thanks to the former proposition.*

We find a decomposition using the free group $\mathbf{F}_2 \simeq [SL(2, \mathbb{Z}), SL(2, \mathbb{Z})]$:

Proposition 3. *Any element $V \in GL(2, \mathbb{Z})$ has a unique decomposition*

$$\begin{aligned} &\pm W(A_0, B_0) O^h W_k(S, T), \\ &W(A_0, B_0) \in \mathbf{F}_2 \simeq [SL(2, \mathbb{Z}), SL(2, \mathbb{Z})], \qquad h \in \{0, 1\}, \\ &W_k(S, T) \in \{\mathbf{1}_2, S, ST, STS, STST, STSTS\} \qquad \text{with} \quad k = 0, 1, \ldots, 5. \end{aligned}$$

The elements of the normal subgroup $SL(2, \mathbb{Z})$ in $GL(2, \mathbb{Z})$ are characterized by $h = 0$.

For the last proposition, we defined in $GL(2, \mathbb{Z})$:

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad O = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The words $W(A_0, B_0)$ are written in a multiplicative way with two generators $A_0 = [(TS)^{-1}, S^{-1}]$ and $B_0 = [S^{-1}, (TS)^{-2}]$ of $\mathbf{F}_2$, thanks to [26, p. 97–98]. In fact, we have a presentation with two generators T and $I = OS$ (see [2]):

$$GL(2, \mathbb{Z}) = \langle I, T^{-1} \mid I^2 = ([T^{-1}, I]T^{-1})^4 = ([T^{-1}, I]T^{-1}I)^2 = \mathbf{1}_2\rangle.$$

The subgroup $\pi'(\mathbf{T}_3)$ is generated by $\pi'(X_0) = T^{-1}IOT^{-1}IOIT^{-1}B_0^{-1}$, $\pi'(Y_0) = IOIOA_0^{-1}TS$, $\pi'(Z_0) = IS$. Moreover [2], the triangle group $\mathbf{T}_3 \simeq \pi'(\mathbf{T}_3)$ is isomorphic to the projective

$$PGL(2, \mathbb{Z}) = \langle \overline{I}, \overline{T}^{-1} \mid \overline{I}^2 = ([\overline{T}^{-1}, \overline{I}]\overline{T}^{-1})^2 = ([\overline{T}^{-1}, \overline{I}]\overline{T}^{-1}\overline{I})^2 = \mathbf{1}\rangle.$$

We can verify that $\mathbf{F}_2 \simeq [PSL(2, \mathbb{Z}), PSL(2, \mathbb{Z})]$ has an index 2 in this last group where we have, with $\mathbf{C}_3$ the cyclic group containing three elements, $\overline{V_1} = [\overline{I}, \overline{T}^{-1}]$ and $\overline{V_2} = [\overline{I}, \overline{T}]$:

$$[PGL(2, \mathbb{Z}), PGL(2, \mathbb{Z})] = \langle \overline{V_1}, \overline{V_2} \mid \overline{V_1}^3 = \overline{V_2}^3 = \mathbf{1}\rangle \simeq \mathbf{C}_3 * \mathbf{C}_3.$$

3 Conformal punctured tori

The conformal punctured tori are easily built with the Poincaré $\mathcal{H}$ half-plane. We use four geodesics of $\mathcal{H}$ referred to as αs, $s\beta$, βp, $p\alpha$, not crossing each other, and with α, s, β, p, on the border of $\mathcal{H}$. Any torus is given by transformations $t_A : \alpha p \to s\beta$, $t_B : \alpha s \to p\beta$. These transformations being given by matrices A and B of $SL(2, \mathbb{R})$ acting on $\mathcal{H}$ as conformal transformations, we can compute with $\alpha < 0$, $\beta > 0$, $c \neq 0$, $c' \neq 0$:

$$A = \begin{bmatrix} c\beta & -c\alpha\beta \\ c & (1/c\beta) - c\alpha \end{bmatrix}, \qquad \text{where} \quad c \neq 0,$$

$$B = \begin{bmatrix} c'\alpha & -c'\alpha\beta \\ c' & (1/c'\alpha) - c'\beta \end{bmatrix}, \qquad \text{where} \quad c' \neq 0,$$

$$A(\alpha) = s, \qquad A(p) = \beta, \qquad B(\beta) = s, \qquad B(p) = \alpha.$$

In $SL(2, \mathbb{R})$, the two matrices A and B generate $G = gp(A, B)$ and define a fuchsian group acting on $\mathcal{H}$, where P is the canonical projection from $SL(2, \mathbb{R})$ to $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm\mathbf{1}_2\}$:

$$\Gamma = PG = G/G \cap \{\pm\mathbf{1}_2\} = gp(P(A), P(B)).$$

The Markoff theory with $A = A_0$, $B = B_0$ is given by $c = \beta = -c' = -\alpha = 1$. For more general cases, we consider the commutator $L = [A, B] = ABA^{-1}B^{-1}$. It contents all the necessary information concerning the associated punctured torus because

$$L(s) = ABA^{-1}B^{-1}(s) = ABA^{-1}(\beta) = AB(p) = A(\alpha) = s.$$

Also $\mathrm{tr}\,(L) = \mathrm{tr}\,([A, B]) \leq -2$ and we get the condition due to Fricke:

$$\mathrm{tr}\,(L) + 2 = \mathrm{tr}\,(A)^2 + \mathrm{tr}\,(B)^2 + \mathrm{tr}\,(AB)^2 - \mathrm{tr}\,(A)\,\mathrm{tr}\,(B)\,\mathrm{tr}\,(AB) \leq 0.$$

The parabolic case defined by the condition $\mathrm{tr}\,(L) = -2$ gives the Markoff equation again, thanks to a factor 3 in the traces which are related by:

$$\mathrm{tr}\,(A)^2 + \mathrm{tr}\,(B)^2 + \mathrm{tr}\,(AB)^2 = \mathrm{tr}\,(A)\,\mathrm{tr}\,(B)\,\mathrm{tr}\,(AB).$$

We now have a parametric representation with $(\lambda, \mu) \in \mathbb{R}^2 \backslash \{(0, 0)\}$, also due to Fricke:

$$\mathrm{tr}\,(A) = \frac{1 + \lambda^2 + \mu^2}{\mu}, \qquad \mathrm{tr}\,(B) = \frac{1 + \lambda^2 + \mu^2}{\lambda}, \qquad \mathrm{tr}\,(AB) = \frac{1 + \lambda^2 + \mu^2}{\lambda\mu},$$

The Markoff theory is obtained with $\lambda = \mu = 1$. Easily:

Proposition 4. *Let (A, B) and (A', B') generate $\Gamma = Pgp(A, B)$ and $\Gamma' = Pgp(A', B')$, two Fricke groups associated to conformal punctured tori, we then have the following equivalence:*

1. *(A, B) et (A', B') are equivalent thanks to an interior automorphism of $GL(2, \mathbb{R})$:*

$$A = DA'D^{-1}, \qquad B = DB'D^{-1}, \qquad \text{where} \quad D \in GL(2, \mathbb{R}).$$

2. *The following two triples are equal:*

$$\Pi(A, B) = (\operatorname{tr}(B^{-1}), \operatorname{tr}(A), \operatorname{tr}(B^{-1}A^{-1})), \quad \Pi(A', B') = (\operatorname{tr}(B'^{-1}), \operatorname{tr}(A'), \operatorname{tr}(B'^{-1}A'^{-1})).$$

3. *The couples (A, B) and (A', B') give the same parameters λ, $\mu \in \mathbb{R}^+$*

$$\lambda = (\operatorname{tr}(A)/\operatorname{tr}(AB)) = (\operatorname{tr}(A')/\operatorname{tr}(A'B')), \qquad \mu = (\operatorname{tr}(B)/\operatorname{tr}(AB)) = (\operatorname{tr}(B')/\operatorname{tr}(A'B')).$$

It is easy to develop a theory of reduction for parabolic tori and to find a link with quaternions. It gives:

Proposition 5. *Any conformal equivalence from a parabolic punctured torus $\mathcal{T}_\Gamma^\bullet$ to itself given by an interior automorphism of $GL(2, \mathbb{R})$ is equal to identity.*

The study of the Laplacian on such surfaces is not so easy [43], though important for physics [29]. The parabolic punctured tori are of the form $\mathcal{H}/\langle A, B, L \mid [A, B]L^{-1} = 1\rangle$. Now if $\mathcal{T}^\bullet$ is the associated topological punctured torus, the conformal structure built on it thanks to A and B is only given [40] by a representation $\rho : \pi_1(\mathcal{T}^\bullet, *) \to SL(2, \mathbb{R})$, where $\pi_1(\mathcal{T}^\bullet, *) \simeq \mathbf{F}_2$ is the Poincaré group of the punctured torus. Introducing the space of all the deformations $\mathcal{R} = \mathcal{R}(\pi_1(\mathcal{T}^\bullet, *), PSL(2, \mathbb{R}))$, and the morphism $\overline{\rho} = P \circ \rho$, we find by this construction all the possible parabolic conformal punctured tori $\mathcal{H}/\overline{\rho}(\pi_1(\mathcal{T}^\bullet, *))$. This approach corresponds to the Teichmüller theory, here specialized to punctured tori. Replacing $PSL(2, \mathbb{R})$ by $PSL(2, \mathbb{C})$, under the former proposition 4, we also have a link with the variety of representations [25] of the group of Poincaré $\pi_1(\mathcal{T}^\bullet, *)$:

$$\rho \in \mathcal{R}(\pi_1(\mathcal{T}^\bullet, *), PSL(2, \mathbb{C})) \to (\operatorname{tr}\rho(g_1), \operatorname{tr}\rho(g_2), \operatorname{tr}\rho(g_3)) \in \mathbb{C}^3.$$

4 Monodromy

A monodromy representation of the group $\pi_1(\mathcal{T}^\bullet, *)$ is a morphism $\rho : \pi_1(\mathcal{T}^\bullet, *) \longrightarrow GL(n, \mathbb{C})$. Its image is the group of monodromy. Such representations are classified with interior automorphisms of $GL(n, \mathbb{C})$. They are considered in Fuchs differential equations (see [46, p. 75] and [16, 24]) as symmetries letting such an equation invariant:

$$\frac{d^n f}{dz^n} + a_1(z)\frac{d^{n-1} f}{dz^{n-1}} + \cdots + a_n(z) f = 0.$$

With $n = 2$ and $\pi_1(\mathcal{T}^\bullet, *) \simeq \mathbf{F}_2$ generated by A et B, the monodromy representations are completely described in [46, p. 80]. The irreducible ones are given thanks to an interior automorphism of $GL(2, \mathbb{C})$ with expressions

$$\rho(A) = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix}, \qquad \rho(B) = \begin{bmatrix} \mu_1 & 0 \\ (\nu_1 + \nu_2) - (\lambda_1\mu_1 + \lambda_2\mu_2) & \mu_2 \end{bmatrix}, \qquad \lambda_i\mu_j \neq \nu_k.$$

They are uniquely determined by the couples (λ_1, λ_2), (μ_1, μ_2), (ν_1, ν_2) of eigenvalues of A, B and AB, with the former constraints. Diagonalizing the matrices A_0 et B_0 of the Markoff theory, we can consider:

$$\rho(A_0) = \begin{bmatrix} \frac{3-\sqrt{5}}{2} & 1 \\ 0 & \frac{3+\sqrt{5}}{2} \end{bmatrix}, \qquad \rho(B_0) = \begin{bmatrix} \frac{3-\sqrt{5}}{2} & 0 \\ -4 & \frac{3+\sqrt{5}}{2} \end{bmatrix}.$$

We now give a solution of the corresponding problem of Riemann–Hilbert, which consists in finding a differential equation having ρ as a monodromy representation. For this we use [46, theorem 4.3.2, p. 85] in order to compute the associated Riemann scheme. This way we find the following fuchsian equation (modified hypergeometric), given with $\sigma_3 = 1-\tau_3 = (\log(\frac{3\pm\sqrt{5}}{2}))/2i\pi$ under the following form:

$$z(1-z)\frac{d^2f}{dz^2} + (1-2z)\frac{df}{dz} - (\sigma_3\tau_3)f = \frac{\log(\frac{3+\sqrt{5}}{2})\log(\frac{3-\sqrt{5}}{2})}{4\pi^2 z(1-z)}f.$$

This equation which constitutes the main innovation of this article can be studied with the methods of [6]. Also, we get a differential operator whose spectral analysis is now to be made:

$$\mathfrak{L} = D^2 + \frac{(1-2z)}{z(1-z)}D - \frac{(\sigma_3\tau_3)4\pi^2 z(1-z) + \log\left(\frac{3+\sqrt{5}}{2}\right)\log\left(\frac{3-\sqrt{5}}{2}\right)}{4\pi^2 z^2(1-z)^2}.$$

The comparison of the corresponding spectrum with the Markoff spectrum must now be made and it will be detailed in a future article. In fact, there are two possibilities for $\mathfrak{L}$ due to the chosen value of σ_3 corresponding to the geometrical phenomenon of Schröder pairs for the same punctured torus, and showing that the two possibilities are linked by an easy transformation. A Hamiltonian interpretation could be important in the present case. It is effective for very important physical equation appearing in Physics (Lamé – that is to say periodical Schrödinger in one dimension [13], sine-Gordon, nonlinear Schrödinger, Korteweg–de Vries, ..., solitons) admitting a Hamiltonian representation, with states in an Hilbert space. It could give a solution for our above project.

5 Conclusion

The comparison with the hypergeometrical approach of Harvey Cohn [7] of the Markoff theory needs to be made. He discovered the link with the following relation between the classical modular function J automorphic for $PSL(2,\mathbb{Z})$ and the Weierstrass function $\wp$:

$$1 - J(\tau) = \wp'^2(z) = 4\wp^3(z) + 1.$$

He explained the link with triples of matrices (A, B, C) associated to the Markoff theory and gave the opportunity to look at a formula assuming an hexagonal symmetry

$$dz = \text{const} \times \frac{dJ}{J^{2/3}(J-1)^{1/2}}.$$

It does not seem to the author of this article that the way between these two formulas has been detailed. The problem is known to be linked to an accessory parameter [5, 21] verifying a Lamé differential equation [47, p. 110]. This question also has a link with the Hilbert's 22^{nd} problem [19]. This famous problem is not yet completely solved [41], even if the Lamé equations are much more studied today [44]. We suggest to get insight in this problem for punctured tori through the former developments. Considering the first of the last two equations and differentiating, we get the former differential relations:

$$-J'(\tau)d\tau = 12\wp^2(z)\wp'(z)dz, \qquad \wp'(z) = (1-J(\tau))^{1/2}, \qquad \wp^2(z) = (J(\tau)/4)^{2/3}.$$

The difficulties for integrating the differential relation between dz and dJ are known [45, p. 85–90], together with the links with the hypergeometric function $F(a, b, c, z)$ solution of the differential equation with two singularities $z = 0$ and $z = 1$, where $z \in \mathbb{C}$:

$$E(a, b, c): \quad z(1-z)\frac{d^2F}{dx^2} + (c - (a+b+1)z)\frac{dF}{dx} - abF = 0.$$

When the parameters a, b, c, are real and c, $c-a-b$, $a-b$, non integers, we find the Schwarz application on $\mathcal{D} = \mathbb{C}\backslash\{]-\infty,0]\cup[1,\infty[\}$:

$$Sch : J \in \mathcal{D} \longrightarrow (F(a,b,c,J) : J^{1-c}F(a+1-c,b+1-c,2-c,J)) \in \mathbf{P}^1(C).$$

The expression of H. Cohn between dz et dJ leads to consider the case $a=(1/3)$, $b=0$, $c=(5/6)$ giving $|1-c|=(1/6)$, $|c-a-b|=(1/2)$, $|a-b|=(1/3)$. These values give confirmation that we are in an euclidian hexagonal crystal case. Also we get the known link with the work of R. Dedekind [10] and his function η. Indeed, we get $dz = w(\tau)^2 d\tau$ with:

$$w(\tau) = \text{const}\,\frac{J'(\tau)^{1/2}}{J(\tau)^{1/3}(1-J(\tau))^{1/4}}.$$

A new hypergeometric equation $E((1/12),(1/12),(2/3))$ appears between w and J. The function η is a square root of w (see [4, p.135] or [28, p.180]), which is known to precisely verify:

$$\eta(\tau)^{24} = \frac{1}{(48\pi^2)^3}\frac{J'(\tau)^6}{J(\tau)^4(1-J(\tau))^3}.$$

The function η has indeed a tight link with the Markoff equation [35].

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