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(Pseudo-)Trace Functions and Modular Invariance of Vertex Operator Algebra

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By including interlocked modules, we showed a modular invariance property of (pseudo) trace functions of modules for a 2-dimensional conformal field theory satisfying C_2-finiteness condition.

1 Introduction

When we encounter a new model of 2-dimensional conformal field theory, we usually try to classify all simple modules. As one of the most popular ways, we define an S-transformation $\mathrm{ch}_{W_i}(-1/\tau)$ of a character

$$\mathrm{ch}_{W_i}(\tau) = \sum_{n\in\mathbb{C}} \dim W_i(n) q^{n-c/24}, \qquad q = \exp\left(2\pi\sqrt{-1}\tau\right)$$

of some known module $W_i = \oplus_{n\in\mathbb{C}} W_i(n)$ and expand it into a power sum of q, where c is a central charge of the model. In the classical cases with only finitely many simple modules, the expansion became a linear sum of other characters, that is,

$$\mathrm{ch}_{W_i}(-1/\tau) = \sum_j a_{ij}\,\mathrm{ch}_{W_j}(\tau), \qquad a_{ij} \in \mathbb{C}.$$

Furthermore, if W_i is a module identified with a chiral algebra, we call such a module "**vertex operator algebra**", then every simple module has appeared in this expansion (as a trace function) and the fusion rules, which are multiplicities N^s_{jk} of modules W_s in the tensor product $W_j \boxtimes W_k$ (or operator product expansion) of two simple modules W_j, W_k,

$$W_j \boxtimes W_k = \sum N^s_{jk} W_s,$$

were determined by these coefficients a_{ij} (Verlinde formula).

These are just phenomena observed in the known models and there is no guarantee for the future, that is, there is no mathematical proof.

Actually, we have faced some models, which were discovered recently, in which an S-transformation

$$\mathrm{ch}_{W_i}(-1/\tau)$$

of character of some module is not a sum of q-powers and it has $\log q$-terms, (for example, see [3]). For example, a triplet model with central charge -2 has eight simple modules $W_1, \ldots, W_8$ and one of their characters is

$$S^1(\tau) = \frac{1}{2}\left(\eta(\tau)^{-1}\theta_{1,2}(\tau) + \eta(\tau)^2\right)$$

and its S-transformation becomes

$$S^1(-1/\tau) = \frac{1}{4}S^3(\tau) - \frac{1}{4}S^4(\tau) - \frac{\sqrt{-1}\tau}{2}\left(S^1(\tau) + S^2(\tau)\right),$$

which has a term with $\log(q) = 2\pi\sqrt{-1}\tau$. Flohr called such a logarithmic term "a generalized character" in [F].

The purpose of my paper is to show that if a chiral algebra (or *vertex operator algebra*) V of 2-dimensional conformal field theory satisfies the following C_2-cofinite condition

$$\dim V/C_2(V) < \infty,$$

where $C_2(V) = \langle v_{-2}u \,|\, v, u \in V\rangle$, then the above logarithmic terms have natural meaning. Namely, every logarithmic term implies an existence of some kind of a module (of course, it is not a direct sum of simple modules) and conversely, the existence of such a module gives a logarithmic term in an S-transformation of some character.

2 Definition of vertex operator algebra

In this section, we will explain a vertex operator algebra which is identified with a chiral algebra. We are not talking about the known models, but the models we will encounter in future. Therefore, we have to give the precise description of the condition we will use.

Let us check the situation. For a given set of fields $\{f_a(z) : a \in A\}$, define a vector space V spanned by the fields. By adding the fields Ta defined by the differential $f_{Ta}(z) = \frac{d}{dz}f_a(z)$, we may assume that V has an action of derivation T. We also assume that one of $f_a(z)$ is a vacuum, that is, $f_{\mathbf{1}}(z) = 1$. With account of the weights of the fields, V becomes a $\mathbb{Z}$-graded vector space

$$V = \oplus_{n\in\mathbb{Z}}V(n).$$

Consider the normal product

$$f_a(z) * f_b(z) = f_a(z)^- f_b(z) + f_b(z)f_a(z)^+,$$

where $f_a(z)^- = \sum_{m\leq -1} a_m z^{-m-1}$ is the generating operator of $f_a(z) = \sum_{m\in\mathbb{Z}} a_m z^{-m-1}$ and $f_a(z)^+ = \sum_{m\geq 0} a_m z^{-m-1}$ denotes the annihilating operator of $f(z)$. Using the normal product and derivation T, we can define infinitely many products $\{*_n * : n \in \mathbb{Z}\}$ on V as follows:

$$\begin{aligned}
&f_a(z)_n f_b(z) = \frac{1}{2\pi\sqrt{-1}}\oint_{x=0}(x-z)^n[f_a(x), f_b(z)] \qquad \text{for} \quad n \geq 0,\\
&f_a(z)_{-1} f_b(z) = f_a(z) * f_b(z), \qquad \text{and}\\
&f_a(z)_{-k} f_b(z) = \frac{1}{k!}\left(\left(\frac{d}{dz}\right)^k f_a(z)^-\right) f_b(z) + f_b(z)\left(\left(\frac{d}{dz}\right)^k f_a(z)^+\right), \qquad \text{for} \quad k \geq 2.
\end{aligned}$$

We will formulate the above setting as follows. We will treat only local fields in this paper, but we can define a similar setting for super-local fields.

A vertex algebra is a triple $V = (V, Y, \mathbf{1})$ consisting of a $\mathbb{Z}$-graded vector space $V = \oplus_{n\in\mathbb{Z}}V(n)$ and a *vertex operator* Y, which assigns an $\mathrm{End}(V)$-valued function in z for each $v \in V$ whose Lorentz expansion has a form

$$Y(v, z) = \sum_{m\in\mathbb{Z}} v_m z^{-m-1}, \qquad v_m \in \mathrm{End}(V)$$

and a special element $\mathbf{1}$ called *vacuum* satisfying the following conditions: $Y(\mathbf{1},z)$ is an identity on V, $\lim_{z\to 0} Y(v,z)\mathbf{1} = v$ and $Y(v,z)w$ has a lower bound in the power of z for any $w \in W$.

Moreover, for any $w_0 \in V^* = \mathrm{Hom}\,(V,\mathbb{C})$, $w \in V$, and the natural paring $\langle V^*, V\rangle$,

$$\begin{aligned}
&\langle w_0, Y(v_1,z_1)Y(v_2,z_2)w\rangle, \qquad |z_1|>|z_2|>0,\\
&\langle w_0, Y(v_2,z_2)Y(v_1,z_1)w\rangle, \qquad |z_2|>|z_1|>0,\\
&\langle w_0, Y(Y(v_1,z_1-z_2)v_2,z_2)w\rangle, \qquad |z_2|>|z_1-z_2|>0
\end{aligned}$$

are expansions of the same rational function with singularities at z_1, z_2, z_1-z_2 at most in the defined regions, respectively. Here End (V) is the set of linear transformations on V.

It is known that the latter condition is equivalent to

$$(z_1-z_2)^N[Y(v_1,z_1),Y(v_2,z_2)] = 0$$

for some integer N by viewing them as formal power series.

Among many vertex algebras, we are interested in the one which has a relation with conformal field theory, that is, it has a representation of Virasoro algebra and some finiteness conditions. We will call it *a vertex operator algebra*.

Namely, it is a vertex algebra $(V, Y, \mathbf{1})$ such that V has no negative weights and all homogeneous spaces are finite-dimensional, that is,

$$V = \oplus_{n=0}^{\infty} V(n) \qquad \text{and} \qquad \dim V(n) < \infty.$$

It also has a special element $\omega \in V(2)$ called *conformal vector* such that a vertex operator $Y(\omega,z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ of ω satisfies

(1) Virasoro algebra relation:

$$[L(m),L(n)] = (n-m)L(n+m) + \delta_{n+m,0}\binom{m+1}{3}\cdot\frac{c}{2},$$

where $c \in \mathbb{C}$ is called central charge of V;

(2) $L(-1)$-derivativity: $Y(L(-1)v,z) = \frac{d}{dz}Y(v,z)$ and

(3) the grading operator $L(0)$: $L(0) = n$ on $V(n)$.

Of course, we also consider modules. The definition of a module is a pair (W, Y^W) of a vector space W and a vertex operator Y^W such that for each $v \in V$, a vertex operator of v on W is an End (W)-valued function $Y^W(v,z)$ whose Lorentz expansion has a form

$$Y^W(v,z) = \sum_{n\in\mathbb{Z}} v_n^W z^{-n-1}, \qquad v_n^W \in \mathrm{End}\,(W)$$

satisfying

$$Y^W(v,z)w \in W((z)) \qquad \text{for all} \quad w \in W$$

such that the set of vertex operators has a structure of vertex operator algebra isomorphic to V (or its homomorphic image) by the normal products. Namely, we can identify $Y^W(v,z)$ as an original field $f_v(z)$ of V.

The best way to classify all simple $\mathbb{Z}_+$-modules is to determine a Zhu algebra $A(V) = V/O(V)$ which is given as a factor space of V. Namely, following [6], V has a product

$$v * u = \mathrm{Res}_x \frac{(1+x)^{\mathrm{wt}\,(v)}}{x} Y(v,x)u$$

for $v \in V_{\mathrm{wt}\, v}$ and $u \in V$, where $\mathrm{wt}\, v$ denotes the weight of v. Set

$$O(V) = \langle \mathrm{Res}_x \frac{(1+x)^{\mathrm{wt}\,(v)}}{x^2} Y(v,x)u \mid v,u \in V \rangle$$

and $A(V) = V/O(V)$. Then Zhu algebra $A(V)$ is an associative algebra with a product $*$. The essential property of the Zhu algebra is that a top module $W(0)$ of an $\mathbb{Z}_+$-graded module $W = \oplus_{m=0}^{\infty} W(m)$ is an $A(V)$-module, and every $A(V)$-module is a top module of some $\mathbb{N}$-graded module.

If we want to study nonsimple modules, the Zhu algebra is not enough. The above concept was naturally extended to n-th graded piece of $\mathbb{N}$-graded modules in [1]. Set

$$O_n(V) = \langle \mathrm{Res}_x \frac{(1+x)^{\mathrm{wt}\,(v)+n}}{x^{2+2n}} Y(v,x)u \mid v,u \in V \rangle$$

and $A_n(V) = V/O_n(V)$. Like $A(V)$, $A_n(V)$ is an associative algebra with a product

$$v *_n u = \sum_{m=0}^{n} \binom{-n}{m} \mathrm{Res}_x \frac{(1+x)^{\mathrm{wt}\,(v)+n}}{x^{n+m+1}} Y(v,x)u$$

and has a property that an n-th (and less) graded piece $W(n)$ of an $\mathbb{N}$-graded module is an $A_n(V)$-module and every $A_n(V)$-module is an n-th (or less) graded piece of a module. $A_n(V)$ is called an n-th Zhu algebra. Another method is to study a Poisson algebra $V/C_2(V)$ which is also given as a factor space of V, where

$$C_2(V) = \langle v_{-2}u \mid v,u \in V \rangle$$

is a subspace of V spanned by elements of the form $v_{-2}u$ with $v,u \in V$. If $\dim V/C_2(V)$ is finite, then V is called C_2-cofinite. Zhu introduced the C_2-cofinite condition in his paper [6] as a technical condition, in order to make the trace function satisfy a differential equations, he called it Condition C.

It was proved by [1] that as long as V is C_2-cofinite, V has only finitely many simple modules. We note that the known models with finitely many simple modules seem to satisfy this condition.

Classically, we have treated only modules $W = \oplus_{m\in\mathbb{C}} W(m)$ on which the grading operator $L(0)$ acts on $W(m)$ as a scalar m (that is, W is a direct sum of eigenspaces of $L(0)$), but we propose to include modules on which W is a direct sum of generalized eigenspaces

$$W(m) = \left\{ w \in W \mid (L(0)-m)^N w = 0 \text{ for some } N \right\}.$$

We note that if it is a simple module, it is a direct sum of eigenspaces of $L(0)$.

When we cover modules on which the grading operator $L(0)$ does not act semisimply (we call such a module "logarithmic"), we have to change the setting a little. For example, a character is usually given by

$$\mathrm{ch}_W(\tau) = \sum_{m\in\mathbb{C}} \dim W(m) q^{m-c/24},$$

which coincides with

$$\mathrm{ch}_W(\tau) = \sum_{m\in\mathbb{C}} \mathrm{tr}_{W(m)} \left(q^{L(0)-c/24} \right).$$

Since we treat a grading operator $L(0)$ which may not be a scalar on $W(m)$, we denote it by a Jordan decomposition

$$L(0) = L^{\mathrm{semi}}(0) + L^{\mathrm{nil}}(0),$$

where $L^{\rm semi}(0)$ is the semisimple part of $L(0)$, that is, $L^{\rm semi}(0)$ acts on a generalized eigenspace $W(m)$ as a scalar m and $L^{\rm nil}(0)$ is the nilpotent part of $L(0)$ and $L^{\rm semi}(0)L^{\rm nil}(0)=L^{\rm nil}(0)L^{\rm semi}(0)$. In this case, we have

$$q^{L(0)} = q^{L^{\rm semi}(0)}q^{L^{\rm nil}(0)} = q^m\sum_{n=0}^{\infty}\frac{1}{n!}\left(2\pi\sqrt{-1}\tau L^{\rm nil}(0)\right)^n \qquad \text{on} \quad W(m)$$

and so we can see a logarithmic term. We note that since $L^{\rm nil}(0)$ is nilpotent, the above sum is a finite sum.

As we showed, if W is a direct sum of generalized eigenspaces of $L(0)$, $q^{L(0)}$ is well-defined. The C_2-cofinite condition has a close connection with this fact.

Theorem 1 ([4]). *The following are equivalent:*
(1) *V is C_2-cofinite.*
(2) *All modules are $\mathbb{Z}_+$-graded.*
(3) *Every module is a direct sum of generalized eigenspaces of $L(0)$.*
(4) $\mathrm{ch}_W(\tau)$ *are well-defined for all modules W.*

Moreover, if V is C_2-cofinite, then all conformal weights are rational numbers [4].

3 Interlocked module

In the previous section we showed that if $L(0)$ has a nonzero nilpotent part $L^{\rm nil}(0)$, then $q^{L(0)}$ has a $\log(q)$-term. On the other hand, it is natural to consider a character of W

$$\mathrm{ch}_W(\tau) = \sum \dim W(m)q^{L(0)-c/24}$$

as one of trace functions

$$\mathrm{tr}_W(v,\tau) = \sum_m\left(\mathrm{tr}_{W(m)}o(v)q^{L(0)-c/24}\right), \qquad v\in V$$

at $v=\mathbf{1}$, where $o(v)$ is the grade-preserving operator of v, for example, if $v\in V_n$, then $o(v)=v_{n-1}$. Actually, from this point of view, Zhu proved that if V satisfies three conditions: C_2-cofinite condition, V has only finitely many simple modules $\{W_1,\dots,W_k\}$ and all modules are completely reducible, then an S-transformation

$$\left(\frac{1}{-\tau}\right)^{\mathrm{wt}\,[v]}\mathrm{tr}_{W_i}(v,-1/\tau)$$

of trace function $\mathrm{tr}_{W_i}(v,\tau)$ is a linear sum of trace functions $\mathrm{tr}_{W_j}(v,\tau)$, that is,

$$\left(\frac{1}{-\tau}\right)^{\mathrm{wt}\,[v]}\mathrm{tr}_{W_i}(v,-1/\tau) = \sum_{j=1}^{k}a_{ij}\,\mathrm{tr}_{W_j}(v,\tau)$$

for some $a_{ij}\in\mathbb{C}$. For the definition of $\mathrm{wt}[v]$, see [6]. We note that if $L(n)v=0$ for $n>0$, then $\mathrm{wt}\,[v]=\mathrm{wt}\,v$. The important thing is that the coefficients a_{ij} do not depend on v.

Unfortunately, if we use only ordinary trace functions, since $L^{\rm nil}(0)$ is a nilpotent operator, $\mathrm{tr}\,o(v)\left(L^{\rm nil}(0)\right)^m=0$ for $m\geq 1$ and so we do not have a logarithmic form in a trace function. In order to preserve a logarithmic form, we have to extend the concept of trace function. So, the first question is:

What is a trace function, or trace?

The answer is that it is just a symmetric function on the algebra consisting of all grade-preserving operators. So, the next question is whether there is another symmetric function or not. The finite dimensional algebras with faithful symmetric functions are called "symmetric algebras" in mathematics, and this concept has a long history in the ring theory, see [5]. The key points are that every trace function and its S-transformed form offer symmetric functions of n-th Zhu algebra and, conversely, we can construct some modules (we will call "interlocked" with a symmetric function) from symmetric functions of n-th Zhu algebra.

Theorem 2 ([4]). *If we have a logarithmic form in an S-transformed trace function, then there is such a symmetric function. Conversely, if there is such a symmetric function, then it defines a new trace function with logarithmic form.*

Let us show an example. The ordinary trace is a symmetric linear function of $\mathrm{End}\,(W(n))$ for a finite-dimensional vector space $W(n)$. It is uniquely determined up to scalar multiple. However, if we consider a subring R of $\mathrm{End}\,(W(n))$, it is possible to have a new symmetric function of R. We will show an example of such rings.

Definition 1. Let W be a V-module. We call W an *interlocked module* if it satisfies the following condition: For any submodule W_1 of W, there is a submodule W_2 such that $W_1 \cong W/W_2$ and $W_2 \cong W/W_1$ as V-modules.

In particular, since $W/J(W) \cong \mathrm{soc}\,(W)$, every grade-preserving operator α of V on $W(n)$ has a form

$$\alpha = \begin{pmatrix} A & C & B \\ O & E & {}^{*}C \\ O & O & A \end{pmatrix},$$

where we have assumed $J(W) \supset \mathrm{soc}\,(W)$ for simplicity and B is a square matrix corresponding to the part $W/J(W) \to \mathrm{soc}\,(W)$. Here a *Jacobson radical* $J(W)$ denotes the intersection of all maximal submodules of W and a *socle* $\mathrm{soc}\,(W)$ of W is the direct sum of all simple submodules of W. For such an endomorphism α, we define a pseudo-trace by

$$\mathrm{pstr}\,(\alpha) = \mathrm{tr}\, B.$$

It is not difficult to check that the above pseudo-trace is a symmetric linear function and we will use it as well as the ordinary trace.

We note that the ordinary trace is one of pseudo-traces. Moreover, as we expected, it leaves a logarithmic term of $L^{\mathrm{nil}}(0)$ and then we can define pseudo-trace function

$$\mathrm{pstr}_W(v,\tau) = \sum_{m\in\mathbb{C}} \mathrm{pstr}_{W(m)}\left(o(v)q^{L^{\mathrm{nil}}(0)}\right) q^{m-c/24}$$

with a logarithmic form.

Furthermore, we have:

Theorem 3 ([4]). *If V is C_2-finite, then the space of pseudo-trace functions:*

$$\langle \mathrm{pstr}_W(v,\tau) \mid W \text{ interlocked mods.}\rangle$$

is $SL_2(\mathbb{Z})$-invariant. Namely, an S-transformation of pseudo-trace function is a linear sum of pseudo-trace functions.

Note that the above space contains all ordinary trace functions and so an S-transformation of ordinary trace function is a linear sum of pseudo-trace functions.

Corollary 1 ([4]). *If $V = \oplus_{m=0}^{\infty} V_m$ is C_2-cofinite and there is no logarithmic modules, then the space spanned by the set of all (ordinary) characters is $SL_2(\mathbb{Z})$-invariant.*

We will also show a bound of the effective central charge $\tilde{c} = c - 24h_{\min}$, where c is the central charge and $h_{\min}$ is the smallest conformal weight.

Corollary 2 ([4]). *If $V = \oplus_{m=0}^{\infty} V_m$ is C_2-cofinite, then the*

$$\tilde{c} \leq \frac{\dim(V/C_2(V)) - 1}{2}.$$

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