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The Structure of Algebras $Q_{n,\vec{\alpha}}$ Generated by Linear Connected Idempotents

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In the article we investigate the structure of algebras generated by linear connected idempotents. In particular, we proof the existence of nonsemisimple ones among such algebras.

Algebras generated by linear connected idempotents were studied in [1–3] and others. In particular, a problem of polynomial relations existence, description of algebras growth $Q_{n,\vec{\alpha}} = \mathbb{C}\langle q_1, q_2, \ldots, q_n \,|\, q_k^2 = q_k;\ \sum\limits_{k=1}^{n} \alpha_k q_k = e\rangle$, $n \in \mathbb{N}$, $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, $\forall\ i$: $\alpha_i \neq 0$, etc. were considered in [2, 3]. All of algebras $Q_{n,\vec{\alpha}}$ are finite-dimensional when $n \leq 3$. Algebras $Q_{n,\vec{\alpha}}$, where $n \geq 4$, are infinite-dimensional for all $\vec{\alpha}$.

But the structure of algebras was studied insufficiently even in the finite-dimensional case (for $n = 3$). In particular, the problem of the existence among such algebras of nonsemisimple ones was not investigated sufficiently.

In this article the algebra

$$Q_{3,\vec{\alpha}} = \mathbb{C}\langle q_1, q_2, q_3 \,|\, q_i^2 = q_i, \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 = e\rangle,$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i > 0$ is studied in detail and here is the proof of the fact that algebras $Q_{3,\vec{\alpha}}$ are nonsemisimple for $\alpha_i = 1$, $\alpha_j + \alpha_k = 1$.

Denote $A = \sum\limits_{i=1}^{3} \alpha_i$ and $M = \{\vec{\alpha} : A = 1,\ A = 2,\ \exists\, i : \alpha_i = 1 \text{ or } \exists\, i, j : \alpha_i + \alpha_j = 1\}$.

Proposition 1. *If $\vec{\alpha} \notin M$ then $Q_{3,\vec{\alpha}} = 0$.*

Proof. Since $\alpha_k q_k = e - (\alpha_i q_i + \alpha_j q_j)$ then $\alpha_k^2 q_k = e + \alpha_i^2 q_i + \alpha_j^2 q_j - 2\alpha_i q_i - 2\alpha_j q_j + \alpha_i \alpha_j \{q_i, q_j\}$. After elimination q_k we obtain $\alpha_k(e - \alpha_i q_i - \alpha_j q_j) = e + \alpha_i^2 q_i + \alpha_j^2 q_j - 2\alpha_i q_i - 2\alpha_j q_j + \alpha_i \alpha_j \{q_i, q_j\}$. That is why $q_j q_i = \frac{1}{\alpha_i \alpha_j}((\alpha_k - 1)e + (2\alpha_i - \alpha_i^2 - \alpha_i \alpha_k) q_i + (2\alpha_j - \alpha_j^2 - \alpha_j \alpha_k) q_j - \alpha_i \alpha_j q_i q_j)$. Since $q_j q_i = q_j(q_j q_i)$ then

$$q_j q_i = \frac{1}{\alpha_i \alpha_j}((\alpha_k - 1 + 2\alpha_j - \alpha_j^2 - \alpha_j \alpha_k) q_j + (2\alpha_i - \alpha_i^2 - \alpha_i \alpha_k) q_j q_i - \alpha_i \alpha_j (q_j q_i) q_j),$$

$$q_j q_i = \frac{1}{\alpha_i \alpha_j}((\alpha_k - 1 + 2\alpha_j - \alpha_j^2 - \alpha_j \alpha_k) q_j + (2\alpha_i - \alpha_i^2 - \alpha_i \alpha_k) q_j q_i$$
$$- ((\alpha_k - 1 + 2\alpha_j - \alpha_j^2 - \alpha_j \alpha_k) q_j + (2\alpha_i - \alpha_i^2 - \alpha_i \alpha_k - \alpha_i \alpha_j) q_i q_j)). \quad (1)$$

From here $(2\alpha_i - \alpha_i^2 - \alpha_i \alpha_k - \alpha_i \alpha_j)[q_i, q_j] = 0$ or $-\alpha_i(\alpha_i + \alpha_j + \alpha_k - 2)[q_i, q_j] = 0$. Since $\vec{\alpha} \notin M$ then $[q_i, q_j] = 0$ and then from (1) it follows $(2\alpha_j - \alpha_j^2 - \alpha_j \alpha_k + \alpha_k - 1 + 2\alpha_i - \alpha_i^2 - \alpha_i \alpha_k - 2\alpha_i \alpha_j) q_i q_j = 0$.

Since $2\alpha_j - \alpha_j^2 - \alpha_j \alpha_k + \alpha_k - 1 + 2\alpha_i - \alpha_i^2 - \alpha_i \alpha_k - 2\alpha_i \alpha_j = -(\alpha_i + \alpha_j - 1)(\alpha_i + \alpha_j + \alpha_k - 1) \neq 0$, when $\alpha \notin M$, then $\forall\ i, j$: $q_i q_j = 0$.

After multiplying $\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 = e$ by q_i we obtain $\alpha_i q_i = q_i$ or $(\alpha_i - 1) q_i = 0$. Since $\vec{\alpha} \notin M$ then $q_i = 0$, $i = 1, 2, 3$. So the algebra is trivial. ■

From the proof follows that the dimension of the algebra $Q_{3,\vec{\alpha}}$ is not more than 4, and if $\alpha_1+\alpha_2+\alpha_3 \neq 2$ then the algebra $Q_{3,\vec{\alpha}}$ is commutative.

Let us prove that $Q_{3,\vec{\alpha}} \neq 0$ if and only if $\vec{\alpha} \in M$ and investigate the structure of algebras $Q_{3,\vec{\alpha}}$, where $\alpha \in M$.

Theorem 1. *Algebra $Q_{3,\vec{\alpha}} \neq 0$ if and only if $\vec{\alpha} \in M$ i.e. one of the conditions 1–9 is satisfied and then*

1) $\alpha_1+\alpha_2+\alpha_3 = 1$ *and the algebra $Q_{3,\vec{\alpha}}$ is one-dimensional algebra;*
2) $\alpha_i = 1$, $\alpha_j \neq 1$, $\alpha_k \neq 1$, $\alpha_j+\alpha_k \neq 1$ *and the algebra $Q_{3,\vec{\alpha}}$ is one-dimensional algebra;*
3) $\alpha_i = \alpha_j = 1$, $\alpha_k \neq 1$ *and the algebra $Q_{3,\vec{\alpha}}$ is commutative two-dimensional semisimple algebra;*
4) $\alpha_1 = \alpha_2 = \alpha_3 = 1$ *and the algebra $Q_{3,\vec{\alpha}}$ is a commutative three-dimensional semisimple algebra;*
5) $\alpha_j + a_k = 1$, $\alpha_i+\alpha_j \neq 1$, $\alpha_i+\alpha_k \neq 1$, $\alpha_i \neq 1$ *and the algebra $Q_{3,\vec{\alpha}}$ is commutative two-dimensional semisimple algebra;*
6) $\alpha_i+\alpha_j = 1$, $\alpha_i+\alpha_k = 1$, $\alpha_j+\alpha_k \neq 1$ *and the algebra $Q_{3,\vec{\alpha}}$ is commutative two-dimensional semisimple algebra;*
7) $\alpha_i+\alpha_j = 1$, $\alpha_i+\alpha_k = 1$, $\alpha_j+\alpha_k = 1$ *that is* $\alpha_i = \alpha_2 = \alpha_3 = \frac{1}{2}$ *and the algebra $Q_{3,\vec{\alpha}}$ is commutative three-dimensional semisimple algebra;*
8) $\alpha_1+\alpha_2+\alpha_3 = 2$ *and* $\forall\ \alpha_i \neq 1$ *and the algebra $Q_{3,\vec{\alpha}}$ is noncommutative four-dimensional semisimple algebra;*
9) $\alpha_i = 1$, $\alpha_j+\alpha_k = 1$ *and the algebra $Q_{3,\vec{\alpha}}$ is noncommutative four-dimensional nonsemisimple algebra, the radical of the algebra is generated by* $q_jq_k - q_j, q_jq_k - q_k$.

Proof. Consider only cases 8 and 9 when algebra is noncommutative.

In the case $\alpha_1+\alpha_2+\alpha_3 = 2$, where $\forall\ \alpha_l \neq 1$, the algebra $Q_{3,\vec{\alpha}}$ is generated by two idempotents q_1 and q_2, so all its irreducible representations are not more than two-dimensional. Since an idempotent in one-dimensional space is equal to 0 or 1 then it is easy to prove that there are no one-dimensional representations of the algebra. Consider an irreducible representation of algebra

$$\pi(q_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi(q_2) = \begin{pmatrix} z & 1 \\ z - z^2 & 1 - z \end{pmatrix}, \quad \pi(q_3) = \begin{pmatrix} \frac{1-\alpha_1-\alpha_2 z}{\alpha_3} & -\frac{\alpha_2}{\alpha_3} \\ \frac{\alpha_2 z^2 - \alpha_2 z}{\alpha_3} & \frac{1+\alpha_2 z - \alpha_2}{\alpha_3} \end{pmatrix},$$

where $z = \frac{(1-\alpha_1)(1-\alpha_2)}{\alpha_2\alpha_1}$.

It is easy to check that from the equation $\pi(x_0 e + x_1 q_1 + x_2 q_2 + x_3 q_1 q_2) = 0$ follows $\forall\ l$: $x_l = 0$. Then 1) since the dimension of the algebra is not more than 4 then elements e, q_1, q_2, q_1q_2 form a basis of the algebra; 2) the representation π is faithful irreducible two-dimensional representation of the algebra. That is why $Q_{3,\vec{\alpha}}$ is a noncommutative four-dimensional semisimple algebra.

Consider now $\alpha_i = 1$, $\alpha_j + \alpha_k = 1$. Take the representation π_1 of algebra:

$$\pi_1(q_i) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \pi_1(q_j) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \pi_1(q_k) = \begin{pmatrix} 0 & -\frac{\alpha_j}{\alpha_k} \\ 0 & 1 \end{pmatrix}.$$

Since π_1 has only one nontrivial invariant subspace then π_1 is not irreducible but indecomposed representation of algebra. A finite-dimensional semisimple algebra has no indecomposed representations which are not irreducible. Then in this case the algebra $Q_{3,\vec{\alpha}}$ is not semisimple.

If $\alpha_i = 1$, $\alpha_j+\alpha_k = 1$ then $q_i+\alpha_j q_j+\alpha_k q_k = e$. After squaring the equality $\alpha_j q_j+\alpha_k q_k = e-q_i$, we obtain

$$\alpha_j^2 q_j + \alpha_k^2 q_k + \alpha_j\alpha_k\{q_j, q_k\} = e - q_i = \alpha_j q_j + \alpha_k q_k.$$

So $(\alpha_j^2 - \alpha_j)q_j + (\alpha_k^2 - \alpha_k)q_k = -\alpha_j\alpha_k\{q_j, q_k\}$.

Since $\alpha_j + \alpha_k = 1$ then $a_j \neq 1$ and $\alpha_k \neq 1$ and $\alpha_j^2 - \alpha_j = -\alpha_j(1-\alpha_j) = -\alpha_j\alpha_k = -(1-\alpha_k)\alpha_k = \alpha_k^2-\alpha_k \neq 0$, so $q_j+q_k = \{q_j, q_k\}$ and $q_k q_j = q_j+q_k-q_j q_k$. We prove that elements e, q_j, q_k, $q_j q_k$ form a basis of the algebra. Suppose in a converse way that there exists some nontrivial linear combination of the elements that is equal to zero: $x_0 e+x_1 q_j+x_2 q_k+x_3 q_j q_k = 0$.

Consider the one-dimensional representation π_0 of the algebra: $\pi_0(q_j) = \pi_0(q_k) = 0$ then $0 = \pi_0(0) = \pi_0(x_0 e + x_1 q_j + x_2 q_k + x_3 q_j q_k) = x_0\pi_0(e) = x_0$ (hereinafter $x_0 = 0$).

Since $\pi_1(q_j q_k) = \pi_1(q_j)$ we have

$$\pi_1(x_0 e + x_1 q_j + x_2 q_k + x_3 q_j q_k) = \begin{pmatrix} 0 & x_1 - \frac{\alpha_j}{\alpha_k}x_2 + x_3 \\ 0 & x_1 + x_2 + x_3 \end{pmatrix} = 0_{2\times 2}. \qquad (2)$$

Define the representation π_2:

$$\pi_2(q_j) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \pi_2(q_k) = \begin{pmatrix} 0 & 0 \\ -\frac{\alpha_j}{\alpha_k} & 1 \end{pmatrix}.$$

Then $\pi_2(q_j q_k) = \pi_2(q_k)$ and

$$\pi_2(x_0 e + x_1 q_j + x_2 q_k + x_3 q_j q_k) = \begin{pmatrix} 0 & 0 \\ x_1 - \frac{\alpha_j}{\alpha_k}(x_2 + x_3) & x_1 + x_2 + x_3 \end{pmatrix} = 0_{2\times 2}. \qquad (3)$$

From (2) and (3) follows that

$$x_1 - \frac{\alpha_j}{\alpha_k}x_2 + x_3 = 0, \qquad x_1 + x_2 + x_3 = 0,$$
$$x_1 - \frac{\alpha_j}{\alpha_k}x_3 = 0, \qquad x_1 - x_3 = 0.$$

From here $\forall\, i$: $x_i = 0$. We have obtained the contradiction. So e, q_j, q_k, $q_j q_k$ form the basis and the dimension of the algebra is equal to 4.

Since the algebra is generated by two idempotents q_j, q_k then dimensions of its irreducible representations are not more than 2. Since irreducible pair of idempotents is equivalent to the pair

$$Q_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q_k = \begin{pmatrix} z & 1 \\ z - z^2 & 1 - z \end{pmatrix},$$

for some $z \in \mathbb{C} \setminus \{0, 1\}$, then one can verify that the algebra has no two-dimensional irreducible representations. So all its irreducible representations are one-dimensional. The algebra has two one-dimensional representations: $\pi_3(q_j) = \pi_3(q_k) = 0$ and $\pi_4(q_j) = \pi_4(q_k) = 1$.

Let $x = x_0 e + x_1 q_j + x_2 q_k + x_3 q_j q_k$ belong to the radical of the algebra then $\pi_3(x) = 0$, $\pi_4(x) = 0$ or $x_0 = 0$, $x_1 + x_2 + x_3 = 0$. The system has a fundamental system of solutions $(0, 1, 0, -1)$, $(0, 0, 1, -1)$. So elements $q_j - q_j q_k$, $q_k - q_j q_k$ generate the radical of algebras. ■

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