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Flow Through a Porous Medium with Multiscale Log-Stable Permeability

Gennady A. KUZ'MIN † *and Olga N. SOBOLEVA* ‡

† *Institute of Thermophysics SB RAS, 630090 Novosibirsk, Russia*
E-mail: *kuzmin@itp.nsc.ru*

‡ *Institute of Computational Mathematics and Mathematical Geophysics SB RAS, 630090 Novosibirsk, Russia*
E-mail: *olga@nmsf.sscc.ru*

We consider single-phase flow of an incompressible fluid through a random scaling porous medium. The joint multi-point probability distribution for the porosity and permeability is supposed to be log-stable and satisfy the conditions of the Kolmogorov's refined scaling hypothesis. A subgrid model is derived which is similar to Landau–Lifschitz formula. The similarity of the present method to the Wilson's renormalization group is noticed.

1 Introduction

We study the multiscale heterogeneity in the subsurface hydrodynamics using the subgrid method similar to that of Wilson's renormalization group. The renormalization group (RG) method was originally developed in the quantum field theory to remove singularities in the perturbation theory. It was used in the variety of problems including the statistical physics, the turbulence theory, certain geophysical problems, and so on (see for example [1]). We use Wilson's formulation of the RG that has some application to subgrid modelling in computational hydrodynamics. Our use of the RG method is motivated by the refined scaling properties [2] that were experimentally observed in the subsurface hydrodynamics [3]. Our previous papers [4–6] describe the general method of analysis and the RG model based on the Kolmogorov's refined scaling to study the subgrid flows. The results conform with those obtained with other formulations of the RG methods (see, for example, [7]). We have considered the logarithmic-normal theory as a simple problem, and the log-stable medium was proposed as a more general task. The present paper is devoted to the flows in the porous self-similar media, described by the selected log-stable probability distributions.

2 The refined scaling of the porous media

Let us consider the scaling hypothesis for the permeability field $\varepsilon(\boldsymbol{x})$. The fluctuations of $\varepsilon(\boldsymbol{x})$ of various sizes may be identified by the spatial smoothing. We use the Fourier filter that omits all the Fourier harmonics whose wavelength is shorter than some threshold value l, when obtaining $\varepsilon_l(\boldsymbol{x})$. At $l \to 0$, $\epsilon_l(\boldsymbol{x}) \to \epsilon(\boldsymbol{x})$. The dimensionless field $\psi(\boldsymbol{x}, l, l') = \epsilon_{l'}(\boldsymbol{x})/\epsilon_l(\boldsymbol{x})$ is similar to the dimensionless fields by Kolmogorov [2]. We assume the similarity hypothesis. Namely, the field $\psi(\boldsymbol{x}, l, l')$ is assumed to be translatory homogeneous, isotropic and of the scaling symmetry. The latter means that $\psi(\boldsymbol{x}, l, l')$ has the same probability distributions as $\psi(K\boldsymbol{x}, Kl, Kl')$, where K a positive number. When introducing the scaling hypothesis, one usually uses the notion of the scale dimension Δ_ψ, that means that the fields $\psi(\boldsymbol{x}, l, l')$ and $K^{\Delta_\psi}\psi(K\boldsymbol{x}, Kl, Kl')$ have the same probability distribution. We suppose that for our nondimensional function ψ $\Delta_\psi = 0$.

The field $\psi(\boldsymbol{x}, l, l')$ has too many arguments. We define a simpler field that contains the same information. To introduce such a field we use the following identity that holds by definition of $\psi(\boldsymbol{x}, l, l')$,

$$\psi(\boldsymbol{x}, l, l'') = \psi(\boldsymbol{x}, l, l')\psi(\boldsymbol{x}, l', l''). \tag{1}$$

Identity (1) results in the exponential representation of ψ and ε (see (3)). The derivation of the representation that is not very rigorous, is the following one.

Let us consider the limit $l'' \to l'$. In the first order in $l''-l'$, we obtain the differential equation:

$$\frac{\partial \psi(\boldsymbol{x}, l, l')}{\partial l'} = \frac{1}{l'}\psi(\boldsymbol{x}, l, l')\varphi(\boldsymbol{x}, l'), \tag{2}$$

where $\varphi(\boldsymbol{x}, l') = \left.\frac{\partial \psi(\boldsymbol{x}, l', l'y)}{l'\partial y}\right|_{y=1}$ is a dimensionless field. From the definition of $\psi(\boldsymbol{x}, l, l')$, we obtain the equation that expresses the field $\psi(\boldsymbol{x}, l, l')$ or $\varepsilon(\boldsymbol{x}, l)$ in terms of $\varphi(\boldsymbol{x}, l)$

$$\frac{\partial \varepsilon_l(\boldsymbol{x})}{\partial \ln l} = \varphi(\boldsymbol{x}, l)\varepsilon_l(\boldsymbol{x}). \tag{3}$$

The self-similar fluctuations may be observed within a finite range of scales $l_0 < l < L$. Equation (3) should be supplemented by the boundary condition on any end of the range (l_η, L). The solution to (3) is the following

$$\varepsilon_l(\boldsymbol{x}) = \varepsilon_0 \exp\left[-\int_l^L \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\right]. \tag{4}$$

The permeability $\epsilon(\boldsymbol{x})$ at the smallest scale l_0 is determined by formula (4) with $l \to l_0$

$$\varepsilon(\boldsymbol{x}) = \varepsilon_0 \exp\left[-\int_{l_0}^L \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\right]. \tag{5}$$

Formula (5) will be used independent of its derivation. It determines the statistical model by defining the statistical distribution for $\varphi(\boldsymbol{x}, l_1)$. When deriving (4) we assumed that the derivative of ψ on l exists. Let us note, however, that a similar but more general exponential solution to (1) is expressed via integral over measure. For simplicity, we use expressions (4), (5).

The scaling models are defined by the field $\varphi(\boldsymbol{x}, l)$ which is assumed to be homogeneous, isotropic and statistically invariant to the scale transform $\varphi(\boldsymbol{x}, l) \to \varphi(K\boldsymbol{x}, Kl)$. The discrete approximation of the fields $\varphi(\boldsymbol{x}, l_1)$ will be considered to render the probabilistic models. In these approximations, the fields $\varphi(\boldsymbol{x}, l)$, $\varphi(\boldsymbol{y}, l')$ with different scales l, l', at any $\boldsymbol{x}$, $\boldsymbol{y}$ are considered to be statistically independent. This supposition is usually assumed in the scaling models and reflects the decay of statistical dependence when the scales of fluctuations become different in the order of magnitude. The latter was proposed in [2].

To describe the probability distribution of $\int_l^L \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}$ for large L/l we use the theorem about sums of independent variables [8, 9]. If the dispersion of $\varphi(\boldsymbol{x}, l)$ at a given point exists, then the theorem says that the factor $\int_{l_0}^L \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}$ for very large L/l_0 tends to a normal field. In the opposite case (the second correlation function does not exist) the integral over $\frac{dl}{l}$ tends to a field described by a stable distribution. For simplicity, the same distribution is assumed for $\varphi(\boldsymbol{x}, l)$, $l_0 < l < L$.

3 Subgrid modeling

We present a subgrid model for a single-phase flow of incompressible fluid through the scale-invariant porous rock with log-stable distribution, keeping in mind that similar methods might be useful for the wave field evolution and for other related characteristics.

The problem is formulated as follows. Let an incompressible fluid steadily flow through a porous medium with the fluctuating permeability coefficient $\epsilon(\boldsymbol{x})$. The simplest model is Darcy's law which expresses the velocity $\boldsymbol{v} = -\epsilon(\boldsymbol{x})\nabla p$ via the pressure p. The incompressibility condition $\operatorname{div} \boldsymbol{v} = 0$ leads to the equation for p:

$$\frac{\partial}{\partial x_j}\left[\varepsilon(\boldsymbol{x})\frac{\partial}{\partial x_j}p(\boldsymbol{x})\right] = 0. \tag{6}$$

An extremely wide range of scales is supposed to exist $L/l_0 \gg 1$. The direct computation of the pressure field $p(\boldsymbol{x})$ from the equation is impossible or requires enormous computational costs. We derive from (6) an equation that describes only the fluctuating pressure field of the largest-scale fluctuations outside the self-similar range.

The spatial smoothing is defined in the previous section for the permeability field. Here we use instead the statistical smoothing of $\varepsilon(\boldsymbol{x})$. The statistical distributions for the fields $\varphi(\boldsymbol{x}, l_1)$ with $l_0 < l_1 < L$ determine a statistical model for the medium. The function $\varepsilon(\boldsymbol{x})$ is divided into two components with respect to the scale l. The large-scale component $\varepsilon(\boldsymbol{x}, l)$ is obtained from $\varepsilon(\boldsymbol{x})$ via statistical average over all $\varphi(\boldsymbol{x}, l_1)$ with $l_1 < l$, $\varepsilon'(\boldsymbol{x}) = \varepsilon(\boldsymbol{x}) - \varepsilon(\boldsymbol{x}, l)$ being a complementary small-scale component

$$\varepsilon(\boldsymbol{x}, l) = \varepsilon_0 \exp\left[-\int_l^L \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\right]\left\langle \exp\left[-\int_{l_0}^l \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\right]\right\rangle_< \tag{7}$$

$$\varepsilon'(\boldsymbol{x}) = \varepsilon(\boldsymbol{x}, l)\left[\frac{\exp\left[-\int_{l_0}^l \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\right]}{\left\langle \exp\left[-\int_{l_0}^l \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\right]\right\rangle_<} - 1\right], \tag{8}$$

where $\langle\cdot\rangle_<$ denotes averaging over $\varphi(\boldsymbol{x}, l_1)$ of small scales l_1, $D = 1, 2, 3\ldots$ is the spatial dimension. The probability distribution for p is determined by ε from equation (6). We define the large-scale (ongrid) pressure field $p(\boldsymbol{x}, l)$ as averaged solution to (6), where the large-scale component $\varepsilon(\boldsymbol{x}, l)$ is fixed, ε' may be random, $p(\boldsymbol{x}, l) = \langle p\,(\boldsymbol{x})\rangle_<$. The complementary subgrid component $p' = p\,(\boldsymbol{x}) - p(\boldsymbol{x}, l)$ cannot generally be rejected from the filtered equation

$$\nabla\left[\varepsilon(\boldsymbol{x}, l)\nabla p_l(\boldsymbol{x}) + \langle\varepsilon'(\boldsymbol{x})\nabla p'(\boldsymbol{x})\rangle_<\right] = 0, \tag{9}$$

because the second term may be essential. The choice of the form of the second term defines the subgrid model.

We use the perturbation theory employing the amplitude of the subgrid fluctuations as a small parameter to evaluate the suitable subgrid model. The actual effective parameter of expansion will be estimated later. In the Wilson's RG, initial l is only slightly greater than the smallest scale l_0. This makes it possible to derive the differential equation in l.

Subtracting (9) from (6), we obtain the equation for the subgrid pressure p'

$$\nabla_i\left[\varepsilon(\boldsymbol{x})\nabla p(\boldsymbol{x})\right] - \nabla_i\left[\varepsilon(\boldsymbol{x}, l)\nabla_i p(\boldsymbol{x}, L) + \langle\varepsilon'(\boldsymbol{x})\nabla_i p'(\boldsymbol{x})\rangle_<\right] = 0. \tag{10}$$

Equation (10) is used to find the subgrid pressure $p'(\boldsymbol{x})$ in terms of the ongrid components. The equation cannot be solved exactly, therefore the perturbation theory is used. The left-hand side of the equation contains the terms of first and second order in the subgrid fluctuations. In this case, we neglect the terms of the second order. To simplify the problem further and to derive the differential equation, let us consider equation (9) in the large-scale limit. This is typical of the Wilson's RG calculations. In the Fourier representation, one considers the zero wavenumber limit in the ongrid expression. The powers of wavenumbers correspond to the gradients of various orders in the spatial representations. The second term in (9) is the average of product of the two subgrid terms. Their aggregate wavenumber may be small if only both factors have almost equal (but oppositely directed) large wavevectors. We conclude that in the above equation, those terms are largest which have a maximum derivative of the subrid terms in the physical space. Thus we arrive at the reduced equation for the subgrid pressure $p'(\boldsymbol{x})$:

$$\Delta p'(\boldsymbol{x}) = -\frac{1}{\varepsilon(\boldsymbol{x}, l)} \nabla\varepsilon'(\boldsymbol{x}) \cdot \nabla p(\boldsymbol{x}, l)$$

that reproduces the Landau–Lifschits model [10]. They estimated the contribution of the random fine-dispersed dielectric permeability to the effective one, that is

$$p'(\boldsymbol{x}) = \int G(\boldsymbol{x} - \boldsymbol{x}') \frac{1}{\varepsilon(\boldsymbol{x}', l)} \nabla_j \varepsilon'(\boldsymbol{x}') \cdot d\boldsymbol{x}' \nabla_j p(\boldsymbol{x}, l),$$

where $G(\boldsymbol{x} - \boldsymbol{x}')$ is the Green's function for the Laplace equation. Again, we have to retain the terms with lower derivatives of the large-scale field $p(\boldsymbol{x}, l)$ treating them as constants in the integrand. Substituting this solution into the filtered equation, we come to the following expression for the subgrid term in (9)

$$\langle\varepsilon'(\boldsymbol{x})\nabla_i p'(\boldsymbol{x})\rangle_< = -\frac{1}{\varepsilon(\boldsymbol{x}, l)} \int \nabla_i G(\boldsymbol{x} - \boldsymbol{x}') \left\langle \varepsilon'(\boldsymbol{x}) \nabla'_j \varepsilon'(\boldsymbol{x}') \right\rangle_< \cdot d\boldsymbol{x}' \nabla_j p(\boldsymbol{x}, l).$$

Integrating by parts and taking into account the isotropy of $\varepsilon'(\boldsymbol{x})$ we obtain

$$\langle\varepsilon'(\boldsymbol{x})\nabla_i p'(\boldsymbol{x})\rangle_< = -\frac{1}{D\varepsilon(\boldsymbol{x}, l)} \left\langle \varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x}) \right\rangle_< \nabla_i p(\boldsymbol{x}, l). \quad (11)$$

This term is of the Landau–Lifschitz type. The subgrid formula is supported by the following direct numerical experiments for the logarithmic-normal scaling distribution (see also [5,6]).

We consider the displacement of a front between two similar fluids while they flow through the nonhomogeneous porous medium. The interface between the fluids are labelled by the particles that passively flow with the fluid. Viscosities and another physical parameters of the fluids are the same. The passive labels are described by the equation

$$m(\boldsymbol{x}) \frac{d\boldsymbol{x}^{(i)}}{dt} = -\varepsilon(\boldsymbol{x}) \nabla p(\boldsymbol{x}).$$

The porosity $m(\boldsymbol{x})$ is modelled similarly to$\varepsilon(\boldsymbol{x})$. We consider various self-similar correlations. An example of probability density is shown in Fig. 1.

Let us now consider the correlation functions of the log-stable permeability. The stable distribution is described by the four indices α, β, γ, λ. The index $0 < \alpha \le 2$ is the most important one. The Gaussian case corresponds to $\alpha = 2$. The other values of α describe various stable distributions. They have the probability distributions but do not have moments of order $n \ge \alpha$. The indices β, γ, λ describe the skewness of the distribution, its center, and scale. Boufadel et al [11] argues that the experimental data about certain sandstones may have the log-stable distributions with $\alpha \approx 1.86$, $\beta = 1$.

Expression (11) contains the second correlation function of the subgrid permeability. Contrary to $\phi(\boldsymbol{x}, l)$, the moments of the function $\varepsilon'(\boldsymbol{x})$ are determined by the exponential function (8) and exist at $\beta = 1$. This assertion uses theorem 2.5.2 [9]. The case $\beta = 1$ at $\alpha < 2$ is the only one when the necessary moments exist. To define the log-stable distributed $\varepsilon(\boldsymbol{x})$ we use the discrete approximation. In formula (4), the integral is replaced by the finite difference formula

$$\varepsilon(\boldsymbol{x}) = \varepsilon_0 2^{\left[-\delta\tau \sum_{i=0}^{N} \varphi(\boldsymbol{x}, \tau_i)\right]}, \quad (12)$$

where $l = 2^\tau$, $\delta\tau = \delta \log_2 l$ is a step in logarithm of scale.

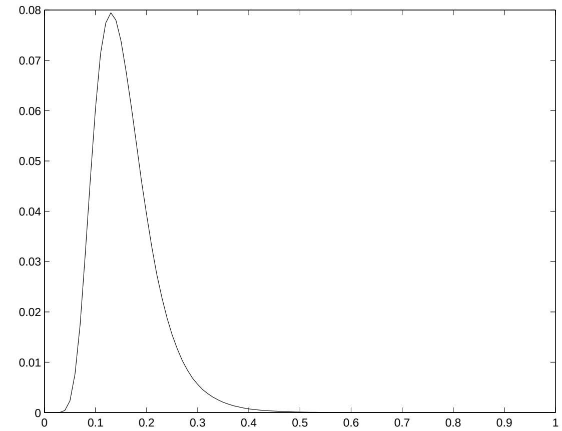

Figure 1. The probability density of porosity.

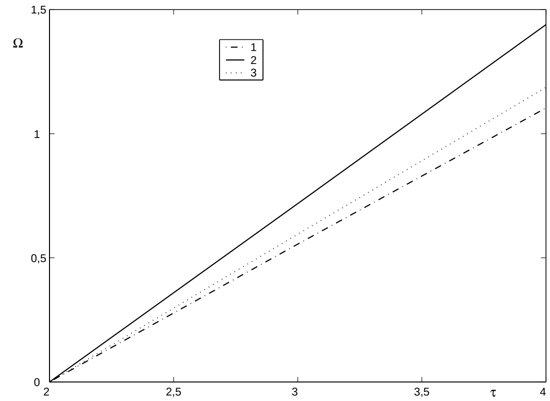

Figure 2. Direct numerical test of the subgrid formula. The flow is along the axis y. $\tau = \log_2 l$, where l is the minimum scale in (4), $\Omega = \mathrm{Log}_2 \langle (dy/dt)/u \rangle$, u is some suitable scale of velocity. 1 – negative correlation of m, ε; 2 – the intermediate positive correlation of m, ε; 3 – the positive correlation of m, ε. The theoretical formula fits the computed values.

We define the stable field $\varphi(\boldsymbol{x}, \tau_i)$ by a simple formula, via a discrete sum of the same independent stable variables ζ_k [12], expression 2.3.6 $\varphi(\boldsymbol{x}_j, \tau) = a_{jk} Y_k$. The sum over repeated indices is assumed. The similarity to the Wiener's formula, which expresses the Gaussian field in terms of the white Gaussian noise is obvious. Here the stable process is expressed in terms of the stable white noise. Average of (5) over subgrid fluctuations of $\varphi(\boldsymbol{x}, \tau)$ leads to products of expressions which are similar to characteristic functions of a stable variable of the form

$$\left\langle \varepsilon_0 2^{\left[-\delta\tau \sum\limits_{l=1}^{N} \varphi(\boldsymbol{x},\tau_l)\right]} \right\rangle_< = \prod_{\boldsymbol{j}}^{l} \left\langle \exp\left(-a^l_{\boldsymbol{i},\boldsymbol{j}} \zeta^l_{\boldsymbol{j}} \delta\tau\right) \right\rangle_< .$$

All the layers are averaged over the statistically independent layers which correspond to different scales, labelled by the index $l = 1, 2, \ldots, N$; $\delta\tau$ is a step in the logarithm of scale, $\boldsymbol{i}$, $\boldsymbol{j}$ correspond to the points of the spatial grid. The kernel $a^l_{\boldsymbol{ij}}$ denotes the spherical and scale-symmetric function $a^l_{\boldsymbol{ij}} \equiv a^l \left(\frac{|\boldsymbol{i}-\boldsymbol{j}|}{l}\right)$. The averages are calculated using the stable distributions contained in [9]. Integrals converge for both $\beta = 1$, $\alpha > 1$ and $\beta = 1$, $\alpha < 1$. The result equals

$$\left\langle \varepsilon_0 2^{\left[-\delta\tau \sum\limits_{l'<l} \varphi(\boldsymbol{x},\ln l')\right]} \right\rangle_< = \exp\left\{ \delta\tau \sum^{l'<l} \lambda \left[-a^{l'}_{\boldsymbol{ij}} \gamma + \left(a^{l'}_{\boldsymbol{ij}}\right)^\alpha \right] \right\}.$$

Taking into account the layers $l' > l$ that are not averaged, we obtain

$$\varepsilon(\boldsymbol{x}, l) = \varepsilon_0 \exp\left[-\sum_{\boldsymbol{j}}^{l'>l} a^{l'}_{\boldsymbol{ij}} \zeta^{l'}_{\boldsymbol{j}} + \left\{ \sum_{\boldsymbol{j}}^{l'<l} \lambda \left[-a^l_{\boldsymbol{ij}} \gamma + \left(a^l_{\boldsymbol{ij}}\right)^\alpha \right] \right\} \right],$$

$$\varepsilon'(\boldsymbol{x}) = \varepsilon(\boldsymbol{x}, l) \left\{ \exp\left[-\sum_{\boldsymbol{j}}^{l'<l} a^{l'}_{\boldsymbol{ij}} \zeta^{l'}_{\boldsymbol{j}} + \lambda \left\{ \sum_{\boldsymbol{j}}^{l'<l} \lambda \left[a^{l'}_{\boldsymbol{ij}} \gamma - \left(a^{l'}_{\boldsymbol{ij}}\right)^\alpha \right] \right\} \right] - 1 \right\}.$$

According to (11), the second moment of the subgrid permeability is needed. Similar direct evaluation yields:

$$\left\langle \varepsilon'(\boldsymbol{x})\, \varepsilon'(\boldsymbol{x}) \right\rangle_< = \varepsilon(\boldsymbol{x}, l)\, \varepsilon(\boldsymbol{x}, l) \times \left\{ \left\langle \exp\left[-2 \sum_{\boldsymbol{j}}^{l'<l} a_{\boldsymbol{ij}}^{l'} \zeta_{\boldsymbol{j}}^{l'} \right] \right\rangle \exp\left[2\lambda \left\{ \widehat{\sum_{\boldsymbol{j}}^{l}} \left[a_{\boldsymbol{ij}}^{l} \gamma - \left(a_{\boldsymbol{ij}}^{l}\right)^{\alpha} \right] \right\} \right] - 1 \right\}.$$

In the first order of the perturbation theory,

$$\left\langle \varepsilon'(\boldsymbol{x})\, \nabla p'(\boldsymbol{x}) \right\rangle_< \approx - \frac{\lambda \sum_{\boldsymbol{j}}^{l'<l} \left[(2^{\alpha} - 2) \left(a_{\boldsymbol{ij}}^{l'} \right)^{\alpha} \right]}{D} \varepsilon(\boldsymbol{x}, l)\, \nabla p(\boldsymbol{x}, l).$$

Substituting this into (9), we can conclude that the ongrid pressure is described by formula (5) in which ε_0 is replaced by the effective constant ε_{0l}. The latter satisfies the equation

$$\frac{d \ln \varepsilon_{0l}}{d \ln l} = \lambda \sum^{l'<l} \left[-a_{ij}^{l'} \gamma + \left(1 - \frac{2^{\alpha} - 2}{D} \right) \left(a_{ij}^{l'} \right)^{\alpha} \right].$$

Let us consider a particular simple case in which the statistical smoothing is most similar to the spatial one

$$\sum_j \left(a_{ij}^{l} \right)^{\alpha} = \gamma, \qquad \sum_j a_{ij}^{l} = 1.$$

In this case, the statistical smoothing does not change the average permeability that is equal to ε_0. However the effective permeability is described by the equation

$$\frac{d \ln \varepsilon_{0l}}{d \ln l} = -\frac{2^{\alpha} - 2}{D} \lambda \gamma.$$

The effective permeability by the main scale L equals to ε_{00}

$$\varepsilon_0 = \varepsilon_{00} \left(\frac{l_0}{L} \right)^{-\frac{2^{\alpha}-2}{D} \lambda \gamma}. \tag{13}$$

Equality (13) is one of the main results of the present paper. It is similar to the Landau–Lifschitz formula but describes the log-stable media. It contains the two constants ε_0 and ε_{00}. The latter describes the mean flow through the media $\overline{\boldsymbol{v}} = \varepsilon_{00} \nabla \overline{p}$. The former is the constant of the self-similar permeability (4). The fact that those constants are different, is the evidence that the subgrid fluctuations of the pressure are essential. It is evident that the effect of fluctuations changes its sign when α crosses 1. On the other hand, it is seen that the power in (13) is small at large D and/or $\alpha \to 1$. This gives the hope that $\frac{2^{\alpha}-2}{D} \lambda \gamma$ may be used as a parameter of expansion in the perturbation theory. To reveal this possibility we consider the terms of higher orders.

Let us again consider equation (10) and substitute $\varepsilon(\boldsymbol{x}) = \varepsilon(\boldsymbol{x}, l) + \varepsilon'(\boldsymbol{x})$. The fluctuations of pressure are the sum of terms of different orders

$$p'(\boldsymbol{x}) = p_1'(\boldsymbol{x}) + p_2'(\boldsymbol{x}) + p_3'(\boldsymbol{x}) + \cdots.$$

We have considered the term of the first order. The terms of the second order yield the equation for $p_2'(\boldsymbol{x})$

$$\nabla_m \left[\varepsilon(\boldsymbol{x}, l) \nabla_m p_2'(\boldsymbol{x}) \right] + \nabla_m \left[\varepsilon'(\boldsymbol{x})\, \nabla_m p_1'(\boldsymbol{x}) - \langle \varepsilon'(\boldsymbol{x}) \nabla_m p_1'(\boldsymbol{x}) \rangle_< \right] = 0. \tag{14}$$

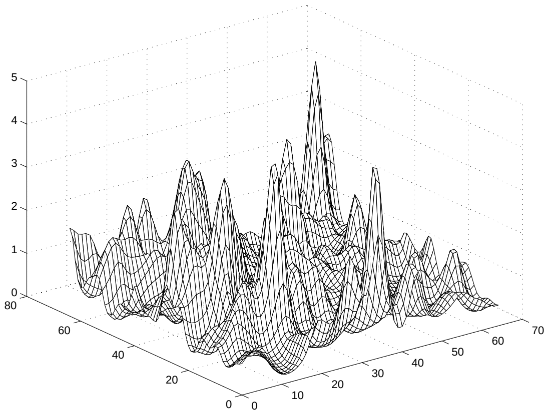

Figure 3. Typical realizations of the logarithmic-normal process.

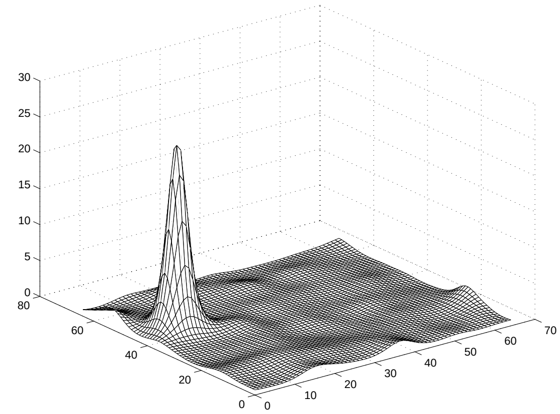

Figure 4. A realization of the log-stable process, $\alpha = 1.86$.

The solution for $p_2'(\boldsymbol{x})$ is obtained as for $p_1'(\boldsymbol{x})$ in the large-scale limit

$$p_2'(\boldsymbol{x}) = -\frac{1}{\varepsilon(\boldsymbol{x},l)^2}\int G(\boldsymbol{x}-\boldsymbol{x}_1) \times \nabla_{1m}\left[\begin{array}{c}\varepsilon'(\boldsymbol{x}_1)\nabla_{1m}\left\{\int d\boldsymbol{x}' G(\boldsymbol{x}_1-\boldsymbol{x}')\nabla_j\varepsilon'(\boldsymbol{x}')\nabla_j p(\boldsymbol{x}_1,l)\right\} \\ -\langle\varepsilon'(\boldsymbol{x}_1)\nabla_{1m}\left\{\int d\boldsymbol{x}' G(\boldsymbol{x}_1-\boldsymbol{x}')\nabla_j\varepsilon'(\boldsymbol{x}')\nabla_j p(\boldsymbol{x}_1,l)\right\}\rangle_< \end{array}\right],$$

When evaluating this integral the third-order correlation function appear which can be calculated in the same manner as above.

$$\begin{aligned}
&\left\langle\varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x})\right\rangle_< \\
&= \varepsilon^3(\boldsymbol{x},l)\left\langle\left\{\exp\left[-\sum_{\boldsymbol{j}}^{l'<l} a_{\boldsymbol{ij}}^{l'}\zeta_{\boldsymbol{j}}^{l'} + \lambda\left\{\sum_{\boldsymbol{j}}^{l'<l}\lambda\left[a_{\boldsymbol{ij}}^{l'}\gamma - \left(a_{\boldsymbol{ij}}^{l'}\right)^\alpha\right]\right\}\right] - 1\right\}^3\right\rangle_< \\
&\approx \varepsilon^3(\boldsymbol{x},l)\left\{\begin{array}{c}\exp\left[(3^\alpha-3)\lambda\left\{\sum_{\boldsymbol{j}}^{l'<l}\lambda\left[\left(a_{\boldsymbol{ij}}^{l'}\right)^\alpha\right]\right\}\right] \\ -3\exp\left[(2^\alpha-2)\lambda\left\{\sum_{\boldsymbol{j}}^{l'<l}\lambda\left[a_{\boldsymbol{ij}}^{l'}\gamma-\left(a_{\boldsymbol{ij}}^{l'}\right)^\alpha\right]\right\}\right]+2\end{array}\right\} \\
&\approx \varepsilon^3(\boldsymbol{x},l)\left\{\begin{array}{c}\exp\left[(3^\alpha-3)\lambda\left\{\sum_{\boldsymbol{j}}^{l'<l}\lambda\left[\left(a_{\boldsymbol{ij}}^{l'}\right)^\alpha\right]\right\}\right] \\ -3\exp\left[(2^\alpha-2)\lambda\left\{\sum_{\boldsymbol{j}}^{l'<l}\lambda\left[a_{\boldsymbol{ij}}^{l'}\gamma-\left(a_{\boldsymbol{ij}}^{l'}\right)^\alpha\right]\right\}\right]+2\end{array}\right\}.
\end{aligned}$$

Obviously, this term is of the next order in the parameter of expansion. The general formula is as follows

$$\left[\varepsilon(\boldsymbol{x},l)\Delta p_{n+1}'(\boldsymbol{x})\right] + \nabla_i\left[\varepsilon'(\boldsymbol{x})\nabla_i p_n'(\boldsymbol{x}) - \langle\varepsilon'(\boldsymbol{x})\nabla_i p_n'(\boldsymbol{x})\rangle_<\right] = 0.$$

Iterating the formula and inserting the result into (9), it is possible to conclude that the high-order terms are smaller in the parameter of expansion. Iterating the formula and inserting the result into (9), one sees that the high-order terms are smaller in the parameter of expansion.

The comparison of the direct numerical simulation with the derived formula is complicated by the following fact. Fig. 3 shows the comparison of the logarithmic-normal ($\alpha = 2$) and Fig. 4 – log-stable (at $\alpha = 1.86$) realizations of permeability.

As is known and confirmed by Fig. 4, the stable processes (and the log-stable) are statistically represented by rare but large events. In order that the statistical characteristics of such processes be found, one needs greater spatial regions. For this reason, the direct numerical test of formula (13) for various values of α will be published in a separate paper.

Acknowledgements

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