Hamiltonian Type Operators in Representations of the Quantum Algebra $su_q(1,1)$

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We study some classes of symmetric operators for the discrete series representations of the quantum algebra $su_q(1,1)$, which may serve as Hamiltonians of various physical systems. These operators are expressed in the canonical basis by a Jacobi matrix. The problem of diagonalization of these operators (eigenfunctions, spectra, overlap coefficients, etc.) is solved by using the connection of such operators with the theory of orthogonal polynomials.

1 Introduction

The theory of representations of the Lie group $SU(1,1) \simeq SL(2,\mathbb{R})$ and its Lie algebra has been extensively employed in various branches of physics and mathematics. Representations of the Lie algebra $su(1,1)$ have been particularly useful in studying the isotropic harmonic oscillator, non-relativistic Coulomb problem, relativistic Schrödinger equation, Dirac equation with the Coulomb interaction, and so on. The Hamiltonian H in the interacting boson model is represented as a linear combination of the operators, corresponding to generating elements J_+^{cl}, J_-^{cl}, J_0^{cl} of the Lie algebra $su(1,1)$. For this reason, the diagonalization of representation operators, corresponding to such linear combinations, is an important problem.

After the appearance of quantum groups and quantum algebras, most problems of the representation theory for Lie groups and Lie algebras have been transferred to the representation theory of quantized groups and algebras. This development is also very important from the point of view of possible applications both in mathematics and in physics. In particular, the diagonalization of representation operators for simplest quantum algebras (especially, such as $su_q(2)$ and $su_q(1,1)$) is of great significance.

Representation operators for the quantum algebras $su_q(2)$ and $su_q(1,1)$ find wide applications in physics. For example, some models in quantum optics, such as Raman and Brillouin scattering, parametric conversion and the interaction of two-level atoms with a single-mode radiation field (Dicke model), can be described by interaction Hamiltonians, which are representation operators for $su_q(2)$ or $su_q(1,1)$ (see, for example, [1] and references therein).

A great interest to spectral analysis of the operators for the positive discrete series of $su_q(1,1)$ appears in the analysis on noncommutative (quantum) spaces. For example, the Laplace operator and the squared radius (together with the third operator, which serves as the operator J_0) generate the algebra $su_q(1,1)$, acting on the space of polynomials on the n-dimensional Manin space or on the quantum complex vector space (see [2–4]). These operators realize irreducible representations of the algebra $su_q(1,1)$, which belong to the positive discrete series and form a q-analogue of the oscillator representations of the Lie algebra $su(1,1)$. To construct Hamiltonians of physical systems, existing in the Manin space or in the quantum complex vector space (for example, Hamiltonians for harmonic oscillators in these spaces), one thus needs to deal with operators of the positive discrete series representations of $su_q(1,1)$. Consequently, the

diagonalization of these Hamiltonians reduces to the problem of diagonalization of representation operators for the quantum algebra $su_q(1,1)$.

In the present paper we discuss the problem of the diagonalization (eigenfunctions, spectra, transition coefficients, etc.) of some classes of operators for the discrete series representations of the quantum algebra $su_q(1,1)$, related to q-orthogonal polynomials. We restrict ourselves by the discrete series representations of $su_q(1,1)$, because these very representation operators are often used as Hamiltonians of physical models and these representations are related to the q-oscillator algebra.

Throughout the sequel we always assume that q is a fixed positive number such that $q < 1$. We extensively use the theory of q-special functions and notations of the standard q-analysis (see, for example, [5] and [6]). In particular, we use q-numbers defined as

$$[a]_q := \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}},$$

where a is any complex number.

2   The algebra $su_q(1,1)$ and its discrete series representations

The classical Lie algebra $su(1,1)$ is generated by the elements J_0^{cl}, J_1^{cl}, J_2^{cl}, satisfying the relations

$$[J_0^{cl}, J_1^{cl}] = i J_2^{cl}, \qquad [J_1^{cl}, J_2^{cl}] = -i J_0^{cl}, \qquad [J_2^{cl}, J_0^{cl}] = i J_1^{cl}.$$

In terms of the raising and lowering operators $J_\pm^{cl} = J_1^{cl} \pm i J_2^{cl}$ these commutation relations can be written as

$$[J_0^{cl}, J_\pm^{cl}] = \pm J_\pm^{cl}, \qquad [J_-^{cl}, J_+^{cl}] = 2 J_0^{cl}.$$

The discrete series representations T_l^+ of $su(1,1)$ with lowest weights are given by a positive number l and they are realized on the spaces $\mathcal{L}_l$ of polynomials in x. The basis in $\mathcal{L}_l$ consists of the monomials

$$g_n^l(x) = \{(2l)_n/n!\}^{1/2} x^n, \qquad n = 0, 1, 2, 3, \dots.$$

Assuming that this basis consists of orthonormal elements, one defines a scalar product in $\mathcal{L}_l$. The closure of $\mathcal{L}_l$ leads to a Hilbert space, on which the representation T_l^+ acts.

We consider an explicit realization of representation operators J_i^{cl}, $i = 0, 1, 2$, in terms of the first-order differential operators:

$$J_0^{cl} = x\frac{d}{dx} + l, \qquad J_1^{cl} = \frac{1}{2}(1 + x^2)\frac{d}{dx} + lx, \qquad J_2^{cl} = \frac{i}{2}(1 - x^2)\frac{d}{dx} - ilx.$$

Then

$$J_0^{cl} g_n^l = (l + n) g_n^l, \qquad J_+^{cl} g_n^l = \sqrt{(2l + n)(n + 1)} \, g_{n+1}^l, \qquad J_-^{cl} g_n^l = \sqrt{(2l + n - 1)n} \, g_{n-1}^l.$$

The quantum algebra $su_q(1,1)$ and its irreducible representations are obtained by deformation of the corresponding relations for the Lie algebra $su(1,1)$ and its irreducible representations. The algebra $su_q(1,1)$ is defined as the associative algebra, generated by the elements J_+, J_-, and J_0, which satisfy the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_-, J_+] = \frac{q^{J_0} - q^{-J_0}}{q^{1/2} - q^{-1/2}} \equiv [2J_0]_q, \qquad (1)$$

and the conjugation relations

$$J_0^* = J_0, \qquad J_+^* = J_-. \tag{2}$$

(Observe that here we have replaced J_- by $-J_-$ in the usual definition of the algebra $\mathrm{sl}_q(2)$.)

We are interested in the discrete series representations of $\mathrm{su}_q(1,1)$ with lowest weights. These irreducible representations will be denoted by T_l^+, where l is a lowest weight, which can be any positive number (see, for example, [7]). These representations are obtained by deforming the corresponding representations of the Lie algebra $\mathrm{su}(1,1)$.

As in the classical case, the representation T_l^+ can be realized on the space $\mathcal{L}_l$ of all polynomials in x. We choose a basis for this space, consisting of the monomials

$$f_n^l \equiv f_n^l(x) := c_n^l \, x^n, \qquad n = 0,1,2,\ldots, \tag{3}$$

where

$$c_0^l = 1, \qquad c_n^l = \prod_{k=1}^{n} \frac{[2l+k-1]_q^{1/2}}{[k]_q^{1/2}} = q^{(1-2l)n/4} \frac{(q^{2l};q)_n^{1/2}}{(q;q)_n^{1/2}}, \qquad n = 1,2,3,\ldots, \tag{4}$$

and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$. The representation T_l^+ is then realized by the operators

$$J_0 = x\frac{d}{dx} + l, \qquad J_\pm = x^{\pm 1}[J_0 \pm l]_q. \tag{5}$$

As a result of this realization, we have

$$J_0 \, f_n^l = (l+n)\, f_n^l, \qquad J_+ \, f_n^l = \sqrt{[2l+n]_q\,[n+1]_q}\, f_{n+1}^l, \tag{6}$$

$$J_- \, f_n^l = \sqrt{[2l+n-1]_q\,[n]_q}\, f_{n-1}^l. \tag{7}$$

Obviously, these operators satisfy the commutation relations (1).

We know that the discrete series representations T_l^+ can be realized on a Hilbert space, on which the conjugation relations (2) are satisfied. In order to obtain such a Hilbert space, we assume that the monomials $f_n^l(x)$, $n = 0,1,2,\ldots$, constitute an orthonormal basis for this Hilbert space. This introduces a scalar product $\langle \cdot, \cdot \rangle$ into the space $\mathcal{L}_l$. Then we close this space with respect to this scalar product and obtain a Hilbert space, which will be denoted by $\mathcal{H}_l$. The Hilbert space $\mathcal{H}_l$ consists of functions (series)

$$f(x) = \sum_{n=0}^{\infty} b_n f_n^l(x) = \sum_{n=0}^{\infty} b_n c_n^l x^n = \sum_{n=0}^{\infty} a_n x^n,$$

where $a_n = b_n c_n^l$. Since $\langle f_m^l, f_n^l \rangle = \delta_{mn}$ by definition, for $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $\tilde{f}(x) = \sum_{n=0}^{\infty} \tilde{a}_n x^n$ we have $\langle f, \tilde{f} \rangle = \sum_{n=0}^{\infty} a_n \tilde{a}_n / |c_n^l|^2$, that is, the Hilbert space $\mathcal{H}_l$ consists of analytical functions $f(x) = \sum_{n=0}^{\infty} a_n x^n$, such that

$$\|f\|^2 \equiv \sum_{n=0}^{\infty} |a_n/c_n^l|^2 < \infty.$$

It is directly checked that for a function $f(x) \in \mathcal{H}_l$ we have $q^{cx\frac{d}{dx}} f(x) = f(q^c x)$. Therefore, taking into account formulas (5), we conclude that

$$q^{J_0/2} f(x) = q^{\frac{1}{2}(x\frac{d}{dx}+l)} f(x) = q^{l/2} f(q^{1/2}x), \tag{8}$$

$$J_+ f(x) = \frac{x}{q^{1/2} - q^{-1/2}} \left[q^l f(q^{1/2}x) - q^{-l} f(q^{-1/2}x) \right], \tag{9}$$

$$J_- f(x) = \frac{1}{(q^{1/2} - q^{-1/2})x} \left[f(q^{1/2}x) - f(q^{-1/2}x) \right]. \tag{10}$$

3   Hamiltonian type operators

We are interested in spectra, eigenfunctions and overlap functions for operators in the representations T_l^+, which correspond to elements of the quantum algebra $su_q(1,1)$ of the form

$$H := q^{p\, J_0}(J_+ + J_-)\, q^{p\, J_0} + f(q^{J_0}), \qquad p \in \mathbb{R}, \tag{11}$$

where f is some polynomial (or function). These operators have the following properties:

  (i) They are representable in the basis (3) by a Jacobi matrix.

 (ii) They are symmetric operators.

(iii) They are not necessarily self-adjoint operators.

Recall that a Jacobi matrix is a matrix, with all entries vanishing except for those which occur on the main diagonal and on two neighbouring (upper and lower) diagonals.

The most important example of the operators of the form (11) is

$$H^{(p)} := q^{pJ_0/4}(J_+ + J_-)q^{pJ_0/4}, \qquad p \in \mathbb{R}.$$

For these operators the following theorem is true [8].

Theorem 1. *If $p > 1$, then the operator $H^{(p)}$ is bounded and has a discrete simple spectrum. Zero is a unique point of accumutation of the spectrum. If $p < 1$, then the closure $\overline{H^{(p)}}$ of the symmetric operator $H^{(p)}$ is not a self-adjoint operator and it has deficiency indices $(1,1)$, that is, $\overline{H^{(p)}}$ has a one-parameter family of self-adjoint extensions. These extensions have discrete simple spectra. If $p = 1$, then $H^{(p)}$ has a continuous simple spectrum, which covers the interval $(-b, b)$, $b = 2/(q^{-1/2} - q^{1/2})$.*

4   Applications of Hamiltonian type operators

There are many applications of the operators (11). Below we exhibit only some of these applications.

4.1. Quantum mechanics in noncommutative world. A big interest in spectral analysis of the operators for the positive discrete series of $su_q(1,1)$ appears in the analysis on noncommutative (quantum) spaces. The n-dimensional Manin space can serve as a simple example of such a space. This space is defined by elements (quantum coordinates) $x_1, x_2, \ldots, x_n$, satisfying the relations $x_i x_j = q x_j x_i$ for $i < j$. A certain q-deformation $U_q'(so_n)$ of the universal enveloping algebra $U(so_n)$ of the Lie algebra so_n (not coinciding with the Drinfeld–Jimbo quantum algebra $U_q(so_n)$) acts on the Manin space (it replaces the action of the rotation group $SO(n)$ on the n-dimensional Euclidean space; for details see [2]).

The squared radius on the Manin space is given by the formula

$$Q = x_1 + q^{-1}x_2^2 + \cdots + q^{-n+1}x_n^2.$$

A Laplace operator on it is defined as

$$\Delta_q = q^{n-1}\partial_1^2 + q^{n-2}\partial_2^2 + \cdots + \partial_n^2$$

(the definition of the derivatives ∂_i are given in [2] and [3]). The operators Δ_q and $\hat{Q}$ (the operator of the multiplication by Q) are invariant with respect to the algebra $U_q'(\mathrm{so}_n)$.

It is known [2] that Δ_q and $\hat{Q}$ generate the quantum algebra $\mathrm{sl}_{q^2}(2)$. Moreover, $J_+ = cQ$ and $J_- = c'\Delta_q$ (where c and c' are constants). In the usual Euclidean space, Hamiltonians of many physical systems are constructed by means of a linear combination of the Laplace operators and the radius. Similarly, Hamiltonians of physical systems in the Manin space are constructed as combinations of the Laplace operator Δ_q and the operator $\hat{Q}$. Then we led to the fact that these Hamiltonians are combinations of the operators J_+ and J_- (see [9]). However, we wish that a Hamiltonian H_q in the Manin space tend to a corresponding Hamiltonian H in the Euclidean space. In order to achieve this, we have to multiply the operator $J_+ + J_-$ from the left and from the right by the operator q^{pJ_0} with an appropriate p. (This multiplication is explained by the fact that q-deforming the classical algebra $U(g)$, where g is a Lie algebra, into the quantum algebra $U_q(g)$, one performs twisting of the elements, corresponding to positive and negative root elements.) Thus, we come to the consideration of Hamiltonians, which are operators of the form (11).

4.2. Quantum optics. Hamiltonians of the type (11) appear in the quantum optics. Examples of such Hamiltonians and further details can be found in [1].

4.3. q-Oscillators, based on $\mathrm{su}_q(2)$ and $\mathrm{su}_q(1,1)$. This q-oscillator, which is different from the q-oscillators of Biedenharn and Macfarlane, is introduced in [10]. It can be described as follows. Let us take in a finite dimensional representation of the algebra $\mathrm{su}_q(2)$ or in a discrete series representation of the algebra $\mathrm{su}_q(1,1)$ the operators

$$B_1 := \frac{1}{2}\, q^{J_0/4}[J_+ + J_-]q^{J_0/4}, \qquad B_2 := \frac{1}{2\mathrm{i}}\, q^{J_0/4}[J_+ - J_-]q^{J_0/4}.$$

Both operators B_1 and B_2 are symmetric and bounded. These operators satisfy the relations

$$[J_0, B_1] = \mathrm{i}B_2, \qquad [J_0, B_2] = \mathrm{i}B_1, \qquad [B_1, B_2] = f(J_0, C), \tag{12}$$

where f is some (explicitly known) function and C is the Casimir operator. Clearly, the operator $f(J_0, C)$ is diagonal in the canonical basis.

Now we introduce a Hamiltonian, a position operator and a momentum operators by the formulas

$$P := -B_2, \qquad Q := B_1, \qquad H = J_0 - l + \frac{1}{2},$$

where l is an index of the representation. Then if f_n^l are basis elements of the representation space, then for the Hamiltonian H we have

$$Hf_n^l = \left(n + \frac{1}{2}\right)f_n^l.$$

It follows from (12) that the position and momentum operators satisfy the relations

$$[H, Q] = -\mathrm{i}P, \qquad [H, P] = \mathrm{i}Q,$$

where $[\cdot, \cdot]$ is the usual commutator.

Thus, using the algebras $\mathrm{su}_q(2)$ and $\mathrm{su}_q(1,1)$, we have obtained Hamiltonian systems of quantum (not q-deformed) mechanics.

4.4. q-Orthogonal polynomials. As will be explained in the next section, operators of the form (11) are closely related to q-orthogonal polynomials. With the aid of these operators we established a notion of duality of polynomials, orthogonal on some countable sets. By means of this duality we derived new orthogonality relations for some instances of polynomials (see [11] and [12]).

5 Relation to moment problem and q-orthogonal polynomials

There exists close relationship between the following directions, which we use for studying Hamiltonian type operators (see [13] and [14]):

(i) the theory of symmetric operators L, representable by a Jacobi matrix;

(ii) the theory of orthogonal polynomials;

(iii) the theory of classical moment problem.

Let us describe this relationship. Let L be a closed symmetric operator on a Hilbert space $\mathcal{H}$. Let $e_1, e_2, \ldots$ be a basis in $\mathcal{H}$ such that

$$Le_n = a_n e_{n+1} + b_n e_n + a_{n-1} e_{n-1}.$$

Let $|x\rangle = \sum\limits_{n=0}^{\infty} p_n(x) e_n$ be an eigenvector of L with the eigenvalue x, that is, $L|x\rangle = x|x\rangle$. Then

$$L|x\rangle = \sum_{n=0}^{\infty} (p_n(x) a_n e_{n+1} + p_n(x) b_n e_n + p_n(x) a_{n-1} e_{n-1}) = x \sum_{n=0}^{\infty} p_n(x) e_n.$$

Equating coefficients of the vector e_n, one comes to a recurrence relation for the coefficients $p_n(x)$:

$$a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x) = x p_n(x).$$

Since $p_{-1}(x) = 0$, then setting $p_0(x) \equiv 1$, we see that this relation completely determines the coefficients $p_n(x)$. Moreover, $p_n(x)$ are polynomials in x of degree n. If coefficients a_n and b_n are real, then all coefficients of the polynomials $p_n(x)$ are real and they are orthogonal with respect to some positive measure $\mu(x)$. If the operator L is self-adjoint, then this measure is uniquely determined and

$$\int p_m(x) p_n(x) d\mu(x) = \delta_{mn},$$

where the integration is taken over some subset (possibly discrete) of $\mathbb{R}$. Moreover, the spectrum of the operator L is simple and coincide with the set, on which the polynomials are orthogonal. The measure $\mu(x)$ determines the spectral measure for the operator L (for details see [14], Chapter VII).

If a closed symmetric operator L is not self-adjoint, then the measure $\mu(x)$ is not determined uniquely. Moreover, in this case there exist infinitely many measures, with respect to which the polynomials are orthogonal. Among these measures there are so-called extremal measures (that is, such that a set of polynomials $\{p_n(x)\}$ is complete in the Hilbert space L^2 with respect to the corresponding measure). These measures determine self-adjoint extensions of the symmetric operator L.

On the other side, with the polynomials $p_n(x)$, $n = 0, 1, 2, \ldots$, the classical moment problem is associated [13]. Namely, with these polynomials (that is, with the coefficients a_n and b_n of the corresponding recurrence relation) positive numbers c_n, $n = 0, 1, 2, \ldots$, are related, which

enter into the classical moment problem. The definition of classical moment problem consists in the following. There is a set of positive numbers c_n, $n = 0, 1, 2, \ldots$. We are looking for a measure $\mu(x)$, such that

$$\int x^n d\mu(x) = c_n, \qquad n = 0, 1, 2, \ldots, \tag{13}$$

where the integration is taken over some fixed subset of $\mathbb{R}$. There are two principal questions:

  (i) Does exist a measure $\mu(x)$, such that relations (13) are satisfied?

 (ii) If such a measure exists, is it determined uniquely?

The answer to the first question is positive, if the numbers c_n, $n = 0, 1, 2, \ldots$, are those, which correspond to a particular family of orthogonal polynomials. Moreover, a measure $\mu(x)$ then coincides with the measure, with respect to which these polynomials are orthogonal.

If a measure in (13) is determined uniquely, then we say that we deal with determined moment problem. It is the case when the region of integration is bounded. If there are many measures, with respect to which relations (13) hold, then we say that we deal with indetermined moment problem. In this case there exist infinitely many measures $\mu(x)$ for which (13) take place. In the second case the corresponding polynomials are orthogonal with respect to all these measures and the corresponding symmetric operator L is not self-adjoint.

Thus, we see that one can study the operator L by investigating the corresponding sets of orthogonal polynomials and their moment problems. We do that here for Hamiltonian type operators.

6 Spectrum and eigenfunctions of Hamiltonians $H^{(\varphi)}$

This section deals with eigenfunctions $\xi_\lambda^l(x; \varphi)$ and eigenvalues of a one-parameter family of the self-adjoint operators

$$H^{(\varphi)} := \frac{1}{2}(q^{1/4} J_+ + q^{-1/4} J_-) q^{J_0/2} + \frac{\cos \varphi}{q^{-1/2} - q^{1/2}} q^{J_0} \tag{14}$$

of the representation T_l^+ of the algebra $\mathrm{su}_q(1, 1)$: $H^{(\varphi)} \xi_\lambda^l(x; \varphi) = \lambda \xi_\lambda^l(x; \varphi)$. Using the relations (8)–(10) we find that

$$H^{(\varphi)} f(x) = c(x^{-1} - 2q^{(2l+1)/4} \cos \varphi + q^{l+1/2} x) f(qx) - c(x^{-1} + q^{1/2-l} x) f(x),$$

where $c = (q^{(2l-1)/4})/2(q^{1/2} - q^{-1/2})$. By using this expression we find (details are given in [15]) that the eigenfunctions of $H^{(\varphi)}$ are

$$\xi_\lambda^l(x; \varphi) = \frac{(axe^{i\varphi}; q)_\infty (axe^{-i\varphi}; q)_\infty}{(bxe^{i(\theta-\varphi)}; q)_\infty (bxe^{i(\varphi-\theta)}; q)_\infty}, \qquad \lambda = \frac{\cos(\theta - \varphi)}{q^{-1/2} - q^{1/2}},$$

where $a = q^{(2l+1)/4}$ and $b = q^{(1-2l)/4}$. A relation between the eigenfunctions $\xi_\lambda^l(x; \varphi)$ and the basis functions $f_n^l(x)$ is now an easy consequence of the generating function

$$\frac{(ae^{i\varphi}t; q)_\infty (ae^{-i\varphi}t; q)_\infty}{(e^{i(\theta-\varphi)}t; q)_\infty (e^{i(\varphi-\theta)}t; q)_\infty} = \sum_0^\infty P_n(\cos(\theta - \varphi); a|q)\, t^n$$

for the q-Meixner–Pollaczek polynomials $P_n(y; a|q)$, defined (see [16], section 3.9) as

$$P_n(\cos(\theta + \varphi); a|q) = a^{-n} e^{-in\varphi} \frac{(a^2; q)_n}{(q; q)_n}\, {}_3\phi_2\left(q^{-n}, ae^{i(\theta+2\varphi)}, ae^{-i\theta}\, a^2, 0\, ; q, q\right).$$

Thus

$$\xi_\lambda^l(x;\varphi) = \sum_{n=0}^{\infty} \frac{q^{(1-2l)n/4}}{c_n^\nu} P_n(\cos(\theta-\varphi);q^l|q) f_n^l(x).$$

To find a spectrum of the operator $H^{(\varphi)}$, we note that the q-Meixner–Pollaczek polynomials $P_n(\cos(\theta-\varphi)) \equiv P_n\left(\cos(\theta-\varphi);q^l|q\right)$ are orthogonal and the orthogonality relation has the form

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_m(\cos(\theta-\varphi)) P_n(\cos(\theta-\varphi)) w_\varphi(\cos(\theta-\varphi)) d\theta = \frac{(q^{2l};q)_n}{(q;q)_n} \delta_{mn},$$

where

$$w_\varphi(\cos(\theta-\varphi)) = (q;q)_\infty (q^{2l};q)_\infty \left| \frac{(e^{2i(\theta-\varphi)};q)_\infty}{q^l(e^{i(\theta-\varphi)};q)_\infty (q^l e^{i\theta};q)_\infty} \right|^2$$

(see formula (3.9.2) in [16]). This orthogonality relation can be written as

$$\int_a^b \frac{(q;q)_n}{(q^{2l};q)_n} P_m(\lambda(q^{-1/2}-q^{1/2})) P_n(\lambda(q^{-1/2}-q^{1/2}))) \hat{w}(\lambda) d\lambda = \delta_{mn},$$

where

$$\hat{w}(\lambda) = \frac{w_\varphi(\lambda(q^{-1/2}-q^{1/2}))(q^{-1/2}-q^{1/2})}{\sin(\varphi-\theta)}, \quad a = \frac{-\cos(\pi+\varphi)}{q^{-1/2}-q^{1/2}}, \quad b = \frac{\cos(\pi-\varphi)}{q^{-1/2}-q^{1/2}}. \quad (15)$$

Therefore, we may formulate the following theorem:

Theorem 2. *The operator $H^{(\varphi)}$ has continuous and simple spectrum, which completely covers the interval (a,b), where a and b are given by* (15).

Continuity of the spectrum means that the eigenfunctions $\xi_\lambda^l(x;\varphi)$ do not belong to the Hilbert space $\mathcal{H}_l$. They belong to the space of functionals on the linear space $\mathcal{L}_l$, which can be considered as a space of generalized functions on $\mathcal{L}_l$.

7 A limit to the classical case

The classical limit (that is, the limit $q \to 1$) has sense only for the operator

$$H^{(\pi/2)} = \frac{1}{2}(q^{1/4} J_+ + q^{-1/4} J_-) q^{J_3/2}.$$

When $q \to 1$ the operator $H^{(\pi/2)}$ tends to the operator $J_1^{(\text{cl})}$: $\lim_{q\to 1} H^{(\pi/2)} = J_1^{(\text{cl})}$. In this limit the basis elements (3) turn into the basis elements $g_n^l(x)$ of the representation space for the Lie algebra su(1,1) and the eigenfunctions $\xi_\lambda^l(x;\pi/2)$ of $H^{(\pi/2)}$ into the eigenfunctions

$$\xi_\lambda^l(x) := (1+ix)^{-l-i\lambda}(1-ix)^{-l+i\lambda}$$

of the operator $J_1^{(\text{cl})}$. They are related to the eigenfunctions of the operator $J_0^{(\text{cl})}$ as

$$\xi_\lambda^l(x) = \sum_{n=0}^{\infty} \left(\frac{n!}{(2l)_n} \right)^{1/2} P_n^{(l)}(\lambda;\pi/2) f_n^l(x) \equiv \sum_{n=0}^{\infty} P_n^{(l)}(\lambda;\pi/2) x^n,$$

where $P_n^{(l)}(\lambda;\pi/2)$ are the classical Meixner–Pollaczek polynomials, defined by the formula

$$P_n^{(l)}(x;\varphi) := \frac{(2l)_n}{n!} e^{in\varphi} {}_2F_1(-n,\nu+ix;2l;1-e^{-2i\varphi}), \qquad l>0, \quad 0<\varphi<\pi.$$

The Meixner–Pollaczek polynomials in the expression for $\xi_\lambda^l(x)$ are a limit case of the corresponding q-Meixner–Pollaczek polynomials (see [16], section 5.9).

8   A limit to the q-oscillator algebra

Let us consider the operator $I := 2H^{(0)}$ from Section 6. It acts upon the canonical basis f_k^l, $k = 0, 1, 2, \ldots$, as

$$I f_k^l = a_k f_{k+1}^l + a_{k-1} f_{k-1}^l + \frac{q^{k+l}}{q^{-1/2} - q^{1/2}} f_k^l, \qquad a_k = \frac{(1 - q^{k+1})^{1/2}(1 - q^{k+2l+2})^{1/2}}{q^{-1/2} - q^{1/2}}.$$

Clearly, the operator I depends on the index l of the representation T_l^+. Taking the limit $l \to +\infty$, we obtain the operator $Q = \lim_{l \to +\infty} (q^{-1} - 1)^{1/2} I$ such that

$$Q f_k = \left(\frac{1 - q^{k+1}}{1 - q} \right)^{1/2} f_{k+1} + \left(\frac{1 - q^k}{1 - q} \right)^{1/2} f_{k-1}.$$

Here f_k is a vector, which must be a limit of the vectors f_k^l as $l \to +\infty$. However, such limit does not exist. In order to have a well-defined limit of basis vectors we change a basis. To this end, we take in $\mathcal{H}_l$ another scalar product. Namely, we take in $\mathcal{L}_l$ the basis consisting of the monomials

$$\tilde{f}_n^l(x) = b_n^l x^n, \quad n = 0, 1, 2, \ldots, \qquad b_0^l = 1, \qquad b_n^l = \frac{(q^{2l}; q)_k^{1/2}}{(q; q)_k^{1/2}}.$$

One can take in $\mathcal{L}_l$ the scalar product such that these vectors constitute an orthonormal basis and close $\mathcal{L}_l$ with respect to this scalar product. The closure of this space will be denoted by $\hat{\mathcal{H}}_l$. Then the operator I is realized by the formula

$$I = \frac{1 - q^{J_0 + l}}{q^{-1/2} - q^{1/2}} x + \frac{1 - q^{J_0 - l}}{q^{-1/2} - q^{1/2}} x^{-1} + \frac{q^{J_0}}{q^{-1/2} - q^{1/2}}$$

and we have

$$I \tilde{f}_k^l(x) = a_k \tilde{f}_{k+1}^l(x) + a_{k-1} \tilde{f}_{k-1}^l(x) + \frac{q^{l+k}}{q^{-1/2} - q^{1/2}} \tilde{f}_k^l,$$

where a_k are the same as above.

Repeating the reasoning of Section 6, we obtain that the eigenfunctions of the operator I in the space $\hat{\mathcal{H}}_l$ are

$$\tilde{\xi}_\lambda^l(x) = \frac{(q^l x; q)_\infty (q^l x; q)_\infty}{(x e^{i\theta}; q)_\infty (x e^{-i\theta}; q)_\infty}, \qquad \lambda = \frac{2 \cos \theta}{q^{-1/2} - q^{1/2}},$$

and these eigenfunctions are decomposed in terms of the basis $\{\tilde{f}_n^l(x)\}$ as

$$\tilde{\xi}_\lambda^l(x) = \sum_{n=0}^{\infty} \frac{1}{b_n^l} P_n(\lambda(q^{-1/2} - q^{1/2}); q^l | q) \tilde{f}_k^l(x).$$

Due to the orthogonality relation for q-Meixner–Pollazcek polynomials, spectrum of the operator I is simple and coincides with the interval $(-a, a)$, $a = 2q^{1/4}(1 - q)^{-1/2}$.

Now we take the limit $l \to +\infty$. Then the basis vectors $\tilde{f}_n^l(x)$ turn into the basis vectors

$$e_n(x) = (q; q)_n^{-1/2} x^n, \qquad n = 0, 1, 2, \ldots.$$

The operator $Q = \lim\limits_{l \to +\infty} (q^{-1} - 1)^{1/2} I$ has the form

$$Qe_n(x) = \left(\frac{1 - q^{n+1}}{1 - q}\right)^{1/2} e_{n+1}(x) + \left(\frac{1 - q^n}{1 - q}\right)^{1/2} e_{n-1}(x) \equiv (a^+ + a_-)e_n(x),$$

where

$$a^+ e_n(x) = \left(\frac{1 - q^{n+1}}{1 - q}\right)^{1/2} e_{n+1}(x), \qquad a^- e_n(x) = \left(\frac{1 - q^n}{1 - q}\right)^{1/2} e_{n-1}(x).$$

The operator $q^{J_0+l} \tilde{f}_k^l(x) = q^k \tilde{f}_k^l(x)$ of the representation T_l^+ of $su_q(1,1)$ turns in the limit as $l \to +\infty$ into the operator

$$q^N e_n(x) = q^n e_n(x).$$

It is easy to see that the operators a^+, a^- and q^N satisfy the relations

$$a^- a^+ - q a^+ a^- = 1, \qquad q^N a^+ = q a^+ q^N, \qquad q^N a^- = q^{-1} a^- q^N,$$

that is, they generate the well-known q-oscillator algebra introduced by Biedenharn and Macfarlane. Clearly, $Q = a^+ + a^-$ is the position operator for this q-oscillator.

The limit $l \to +\infty$ turns the eigenfunctions $\tilde{\xi}_\lambda^l(x)$ into eigenfunctions of the operator Q having the form

$$\tilde{\xi}_\lambda(x) = \frac{1}{(xe^{i\theta}; q)_\infty (xe^{-i\theta}; q)_\infty}, \qquad \lambda = \frac{2\cos\theta}{(1 - q)^{1/2}}.$$

We have $Q\tilde{\xi}_\lambda(x) = \lambda \tilde{\xi}_\lambda(x)$ and a spectrum of the operator Q is simple and coincides with the interval $(-b, b)$, $b = 2/(1 - q)^{1/2}$.

9   Hamiltonian with bounded discrete spectrum

In this section we consider the operator

$$H_1 = q^{3J_0/4}(J_+ + J_-)q^{3J_0/4} - \left([J_0 - l]_q\, q^{l/2} + [J_0 + l]_q\, q^{-l/2}\right) q^{3J_0/2},$$

(note that this operator depends on the index l of the representation T_l^+). It acts on the basis elements (3) by the formula

$$H_1\, f_k^l = a_{k+1} f_{k+1}^l + a_k f_{k-1}^l - q^{3(l+k)/2} d_k f_k^l,$$

where

$$a_k = q^{3(l+k)/2-3/4} \sqrt{[k]_q\, [2l + k - 1]_q}, \qquad d_k = [k]_q\, q^{(l-1)/2} + [2l + k]_q\, q^{-(l-1)/2}.$$

By using this action it is easy to check that the H_1 is a bounded self-adjoint operator. Eigenfunctions of the operator H_1, $H_1\chi_\lambda^l(x) = \lambda\,\chi_\lambda^l(x)$, can be represented in the form

$$\chi_\lambda^l(x) = \sum_{k=0}^\infty P_k(\lambda) f_k^l(x).$$

It is not hard to prove that

$$P_k(\lambda) = \left(\frac{(q^{2l}; q)_k}{(q; q)_k}\right)^{1/2} q^{-lk} p_k(q^y; q^{2l-1}|q), \qquad q^y = (1 - q^{-1})\lambda,$$

where $p_k(q^y; q^{2l-1}|q) = {}_2\phi_1(q^{-k}, 0; q^{2l}; q; q^{y+1})$ are the so-called little q-Laguerre (Wall) polynomials (see, for example, [16], section 3.20).

Due to the orthogonality relation

$$(q^{2l}; q)_\infty \sum_{k=0}^{\infty} \frac{q^{2lk}}{(q; q)_k} p_m(q^k; q^{2l-1}|q) \, p_n(q^k; q^{2l-1}|q) = \frac{q^{2ln}(q; q)_n}{(q^{2l}; q)_n} \delta_{mn}$$

for little q-Laguerre polynomials (see formula (3.20.2) in [16]), a spectrum of the operator H_1 coincides with the set of points $q^n/(1 - q^{-1})$, $n = 0, 1, 2, \ldots$. This means that the eigenfunctions

$$\chi^l_{\lambda_n}(x; q) \equiv \Xi^l_n(x), \qquad n = 0, 1, 2, \ldots, \qquad \lambda_n = \frac{q^n}{1 - q^{-1}},$$

constitute a basis in the representation space. We thus proved the following theorem.

Theorem 3. *The operator H_1 has a simple discrete spectrum, which consists of the points $q^n/(1 - q^{-1})$, $n = 0, 1, 2, \ldots$. The corresponding eigenfunctions $\Xi^l_n(x)$ constitute an orthogonal basis in the space $\mathcal{H}_l$.*

10 Hamiltonian related to big q-Laguerre polynomials

In this section we are interested in the operator

$$H_2 := \alpha q^{J_0/4}(\sqrt{1 - bq^{J_0-l}}J_+ q^{(J_0-l)/2} + q^{(J_0-l)/2}J_-\sqrt{1 - bq^{J_0-l}})q^{J_0/4} - \beta_1 q^{2J_0} + \beta_2 q^{J_0-l}$$

of the representation T_l^+, where $b < 0$ and

$$\alpha = (-b)^{1/2}q^l(1 - q), \qquad \beta_1 = b(1 + q), \qquad \beta_2 = bq + q^{2l}(b + 1).$$

Since the bounded operator q^{J_0} is diagonal in the canonical basis (3) without zero diagonal elements, the operator H_2 is well defined.

We have the following expression for the symmetric operator H_2 in the canonical basis (3):

$$H_2 f^l_n = (-ab)^{1/2}q^{(n+2)/2} \left[\sqrt{(1 - q^{n+1})(1 - aq^{n+1})(1 - bq^{n+1})}f^l_{n+1} \right.$$

$$\left. + q^{-1/2}\sqrt{(1 - q^n)(1 - aq^n)(1 - bq^n)}f^l_{n-1} \right] - [abq^{2n+1}(1 + q) - q^{n+1}(a + ab + b)]f^l_n,$$

where $a = q^{2l-1}$. Since $q < 1$ the operator H_2 is bounded. Therefore, one can close this operator, so we assume that H_2 is a closed (and, consequently, defined on the whole space $\mathcal{H}_l$) operator. Since H_2 is symmetric, its closure is a self-adjoint operator.

For eigenfunctions $\xi_\lambda(x)$ of the operator H_2, $H_2\xi_\lambda(x) = \lambda\xi_\lambda(x)$, we have the expression

$$\xi_\lambda(x) = \sum_{n=0}^{\infty}(-ab)^{-n/2}q^{-n(n+3)/4}\left(\frac{(aq, bq; q)_n}{(q; q)_n}\right)^{1/2} P_n(\lambda; a, b; q)f^l_n(x),$$

where $P_n(\lambda; a, b; q)$ are big q-Laguerre polynomials, defined by the formula

$$P_n(\lambda; a, b; q) := {}_3\phi_2(q^{-n}, 0, \lambda; aq, bq; q, q) = (q^{-n}/b; q)_n^{-1} {}_2\phi_1(q^{-n}, aq/\lambda; aq; q, \lambda/b).$$

The orthogonality relation for these polynomials is known to be of the form

$$\sum_{n=0}^{\infty} \frac{(q^{n+1}; q)_\infty(aq^{n+1}/b; q)_\infty}{(aq^{n+1}; q)_\infty} q^n P_m(aq^{n+1}; a, b; q)P_{m'}(aq^{n+1}; a, b; q)$$

$$- \frac{b}{a}\sum_{n=0}^{\infty} \frac{(q^{n+1}; q)_\infty(bq^{n+1}/a; q)_\infty}{(bq^{n+1}; q)_\infty} q^n P_m(bq^{n+1}; a, b; q)P_{m'}(bq^{n+1}; a, b; q)$$

$$= \frac{(q, b/a, aq/b; q)_\infty}{(aq, bq; q)_\infty} \frac{(q; q)_m}{(aq, bq; q)_m}(-ab)^m q^{m(m+3)/2}\delta_{mm'}.$$

Therefore, we arrive to the following assertion:

Theorem 4. *The spectrum of the operator H_2 coincides with the set of points aq^{n+1}, bq^{n+1}, $n = 0, 1, 2, \ldots$. The spectrum is simple and has only one accumulation point at 0.*

Acknowledgements

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