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p-Mechanics and De Donder–Weyl Theory

Vladimir V. KISIL

School of Mathematics, University of Leeds, Leeds LS2 9JT, UK
On leave from the Odesa University (Ukraine)
E-mail: *kisilv@maths.leeds.ac.uk*
URL: http://amsta.leeds.ac.uk/~kisilv/

The orbit method of Kirillov is used to derive the p-mechanical brackets [26]. They generate the quantum (Moyal) and classic (Poisson) brackets on respective orbits corresponding to representations of the Heisenberg group. The extension of p-mechanics to field theory is made through the De Donder–Weyl Hamiltonian formulation. The principal step is the substitution of the Heisenberg group with Galilean.

1 Introduction

The purpose of this paper is to extended the p-mechanical approach [23,26,25,4,5] from particle mechanics to the field theory. We start in Section 2.1 from the Heisenberg group and its representations derived through the orbit method of Kirillov. In Section 2.2 we define p-mechanical observables as convolutions on the Heisenberg group $\mathbb{H}^n$ and study their commutators. We modify the commutator of two p-observables by the antiderivative to the central vector field in the Heisenberg Lie algebra in Section 2.3, this produces p-mechanical brackets and corresponding dynamic equation. Then the p-mechanical construction is extended to the De Donder–Weyl Hamiltonian formulation of the field theory [12–20] in Section 3.1. To this end we replace the Heisenberg group by the Galilean group in Section 3.2. Expanded presentation of Section 2 could be found in [25]. Development of material from Section 3 will follow in subsequent papers.

2 Elements of p-mechanics

2.1 The Heisenberg group and its representations

Let (s,x,y), where x, $y \in \mathbb{R}^n$ and $s \in \mathbb{R}$, be an element of the Heisenberg group $\mathbb{H}^n$ [9,11]. The group law on $\mathbb{H}^n$ is given as follows:

$$(s,x,y) * (s',x',y') = \left(s + s' + \frac{1}{2}\omega(x,y;x',y'), x+x', y+y' \right), \tag{1}$$

where the non-commutativity is made by ω – the *symplectic form* [1, § 37] on $\mathbb{R}^{2n}$:

$$\omega(x,y;x',y') = xy' - x'y. \tag{2}$$

The Lie algebra $\mathfrak{h}^n$ of $\mathbb{H}^n$ is spanned by left-invariant vector fields

$$S = \partial_s, \qquad X_j = \partial_{x_j} - y_j/2\partial s, \qquad Y_j = \partial_{y_j} + x_j/2\partial s \tag{3}$$

on $\mathbb{H}^n$ with the Heisenberg *commutator relations*

$$[X_i, Y_j] = \delta_{i,j} S \tag{4}$$

and all other commutators vanishing. The exponential map $\exp : \mathfrak{h}^n \to \mathbb{H}^n$ respecting the multiplication (1) and Heisenberg commutators is

$$\exp : sS + \sum_{k=1}^{n}(x_k X_k + y_k Y_k) \mapsto (s, x_1, \ldots, x_n, y_1, \ldots, y_n).$$

As any group $\mathbb{H}^n$ acts on itself by the conjugation automorphisms $\mathrm{A}\,(g)h = g^{-1}hg$, which fix the unit $e \in \mathbb{H}^n$. The differential $\mathrm{Ad}\, : \mathfrak{h}^n \to \mathfrak{h}^n$ of A at e is a linear map which could be differentiated again to the representation ad of the Lie algebra $\mathfrak{h}^n$ by the commutator: $\mathrm{ad}\,(A) : B \mapsto [B, A]$. The dual space $\mathfrak{h}_n^*$ to the Lie algebra $\mathfrak{h}^n$ is realised by the left invariant first order differential forms on $\mathbb{H}^n$. By the duality between $\mathfrak{h}^n$ and $\mathfrak{h}_n^*$ the map Ad generates the *co-adjoint representation* [22, § 15.1] $\mathrm{Ad}^* : \mathfrak{h}_n^* \to \mathfrak{h}_n^*$:

$$\mathrm{ad}^*(s, x, y) : (h, q, p) \mapsto (h, q + hy, p - hx), \qquad \text{where} \quad (s, x, y) \in \mathbb{H}^n \tag{5}$$

and $(h, q, p) \in \mathfrak{h}_n^*$ in bi-orthonormal coordinates to the exponential ones on $\mathfrak{h}^n$. There are two types of orbits in (5) for Ad^* – Euclidean spaces $\mathbb{R}^{2n}$ and single points:

$$\mathcal{O}_h = \{(h, q, p) : \text{ for a fixed } h \neq 0 \text{ and all } (q, p) \in \mathbb{R}^{2n}\}, \tag{6}$$
$$\mathcal{O}_{(q,p)} = \{(0, q, p) : \text{ for a fixed } (q, p) \in \mathbb{R}^{2n}\}. \tag{7}$$

The *orbit method* of Kirillov [22, § 15] starts from the observation that the above orbits parametrise irreducible unitary representations of $\mathbb{H}^n$. All representations are *induced* [22, § 13] by a character $\chi_h(s, 0, 0) = e^{2\pi \mathrm{i} hs}$ of the centre of $\mathbb{H}^n$ generated by $(h, 0, 0) \in \mathfrak{h}_n^*$ and shifts (5) from the *left* on orbits. Using [22, § 13.2, Prob. 5] we get a neat formula, which (unlike many other in literature) respects all *physical units* [25]:

$$\rho_h(s, x, y) : f_h(q, p) \mapsto e^{-2\pi \mathrm{i}(hs + qx + py)} f_h\left(q - \frac{h}{2}y, p + \frac{h}{2}x\right). \tag{8}$$

The derived representation $d\rho_h$ of the Lie algebra $\mathfrak{h}^n$ defined on the vector fields (3) is:

$$d\rho_h(S) = -2\pi \mathrm{i} hI, \qquad d\rho_h(X_j) = h\partial_{p_j} + \frac{\mathrm{i}}{2}q_j I, \qquad d\rho_h(Y_j) = -h\partial_{q_j} + \frac{\mathrm{i}}{2}p_j I. \tag{9}$$

Operators D_h^j, $1 \leq j \leq n$ representing vectors from the complexification of $\mathfrak{h}^n$:

$$D_h^j = d\rho_h(-X_j + \mathrm{i}Y_j) = \frac{h}{2}(\partial_{p_j} + \mathrm{i}\partial_{q_j}) + 2\pi(p_j + \mathrm{i}q_j)I = h\partial_{\bar{z}_j} + 2\pi z_j I, \tag{10}$$

where $z_j = p_j + \mathrm{i}q_j$ are used to give the following classic result in terms of orbits:

Theorem 1 (Stone–von Neumann, cf. [22, § 18.4], [9, Chap. 1, § 5]). *All unitary irreducible representations of* $\mathbb{H}^n$ *are parametrised up to equivalence by two classes of orbits* (6) *and* (7) *of co-adjoint representation* (5) *in* $\mathfrak{h}_n^*$*:*

1. *The infinite dimensional representations by transformation* ρ_h (8) *for* $h \neq 0$ *in Fock* [9,11] *space* $F_2(\mathcal{O}_h) \subset L_2(\mathcal{O}_h)$ *of null solutions to the operators* D_h^j (10)*:*

$$F_2(\mathcal{O}_h) = \{f_h(p, q) \in L_2(\mathcal{O}_h) \mid D_h^j f_h = 0,\ 1 \leq j \leq n\}. \tag{11}$$

2. *The one-dimensional representations as multiplication by a constant on* $\mathbb{C} = L_2(\mathcal{O}_{(q,p)})$ *which drops out from* (8) *for* $h = 0$*:*

$$\rho_{(q,p)}(s, x, y) : c \mapsto e^{-2\pi \mathrm{i}(qx + py)}c. \tag{12}$$

Note that $f_h(p,q)$ is in $F_2(\mathcal{O}_h)$ if and only if the function $f_h(z)e^{-|z|^2/h}$, $z = p + \mathrm{i}q$ is in the classical Segal–Bargmann space [9, 11], particularly is analytical in z. Furthermore the space $F_2(\mathcal{O}_h)$ is spanned by the Gaussian *vacuum vector* $e^{-2\pi(q^2+p^2)/h}$ and all *coherent states*, which are "shifts" of the vacuum vector by operators (8).

Commutative representations (12) correspond to the case $h = 0$ in the formula (8). They are always neglected, however their union naturally (see the appearance of Poisson brackets in (21)) act as the classic *phase space*:

$$\mathcal{O}_0 = \bigcup_{(q,p)\in\mathbb{R}^{2n}} \mathcal{O}_{(q,p)}. \tag{13}$$

Furthermore the structure of orbits of $\mathfrak{h}_n^*$ echoes in equation (22) and its symplectic invariance [25].

2.2 Convolution algebra of $\mathbb{H}^n$ and commutator

Using a left invariant measure dg on $\mathbb{H}^n$ the linear space $L_1(\mathbb{H}^n, dg)$ can be upgraded to an algebra with the convolution multiplication:

$$(k_1 * k_2)(g) = \int_{\mathbb{H}^n} k_1(g_1)\, k_2(g_1^{-1}g)\, dg_1 = \int_{\mathbb{H}^n} k_1(gg_1^{-1})\, k_2(g_1)\, dg_1. \tag{14}$$

Inner *derivations* D_k, $k \in L_1(\mathbb{H}^n)$ of $L_1(\mathbb{H}^n)$ are given by the *commutator* for $f \in L_1(\mathbb{H}^n)$:

$$D_k : f \mapsto [k, f] = k * f - f * k = \int_{\mathbb{H}^n} k(g_1) \left(f\left(g_1^{-1}g\right) - f\left(gg_1^{-1}\right)\right) dg_1. \tag{15}$$

A unitary representation ρ_h of $\mathbb{H}^n$ extends to $L_1(\mathbb{H}^n, dg)$ by the formula:

$$\begin{aligned}
[\rho_h(k)f](q,p) &= \int_{\mathbb{H}^n} k(g)\rho_h(g)f(q,p)\, dg \\
&= \int_{\mathbb{R}^{2n}} \left(\int_{\mathbb{R}} k(s,x,y)e^{-2\pi \mathrm{i} hs}\, ds \right) e^{-2\pi\mathrm{i}(qx+py)} f(q-hy, p+hx)\, dx\, dy,
\end{aligned} \tag{16}$$

thus $\rho_h(k)$ for a fixed $h \neq 0$ depends only from $\hat{k}_s(h,x,y)$ – the partial Fourier transform $s \to h$ of $k(s,x,y)$. Then the representation of the composition of two convolutions depends only from

$$(k' * k)\hat{}_s(h,x,y) = \int_{\mathbb{R}^{2n}} e^{\pi\mathrm{i}h(xy'-yx')}\, \hat{k}'_s(h,x',y')\, \hat{k}_s(h, x-x', y-y')\, dx'dy'.$$

The last expression for the full Fourier transforms of k' and k turn to be the *star product* known in deformation quantisation, cf. [30, (9)–(13)]. Consequently the representation of commutator (15) depends only from [25]:

$$[k', k]\hat{}_s = 2\mathrm{i} \int_{\mathbb{R}^{2n}} \sin(\pi h(xy' - yx'))\, \hat{k}'_s(h,x',y')\hat{k}_s(h, x-x', y-y')\, dx'dy', \tag{17}$$

which turn to be exactly the "Moyal brackets" [30] for the full Fourier transforms of k' and k. Also the expression (17) vanishes for $h = 0$ as can be expected from the commutativity of representations (12).

2.3 p-mechanical brackets on $\mathbb{H}^n$

A multiple $\mathcal{A}$ of a right inverse operator to the vector field S (3) on $\mathbb{H}^n$ is defined by:

$$S\mathcal{A} = 4\pi^2 I, \qquad \text{where} \qquad \mathcal{A}e^{2\pi i hs} = \begin{cases} \frac{2\pi}{ih} e^{2\pi i hs}, & \text{if } h \neq 0, \\ 4\pi^2 s, & \text{if } h = 0. \end{cases} \tag{18}$$

It can be extended by the linearity to $L_1(\mathbb{H}^n)$. We introduce [26] a modified convolution operation $\star$ on $L_1(\mathbb{H}^n)$ and the associated modified commutator:

$$k_1 \star k_2 = k_1 * (\mathcal{A}k_2), \qquad \{[k_1, k_2]\} = k_1 \star k_2 - k_2 \star k_1. \tag{19}$$

Then from (16) one gets $\rho_h(\mathcal{A}k) = (ih)^{-1}\rho_h(k)$ for $h \neq 0$. Consequently the modification of (17) for $h \neq 0$ is only slightly different from the original one:

$$\{[k', k]\}\hat{_s} = \int_{\mathbb{R}^{2n}} \frac{2\pi}{h} \sin(\pi h(xy' - yx'))\, \hat{k}'_s(h, x', y')\, \hat{k}_s(h, x - x', y - y')\, dx'dy', \tag{20}$$

However the last expression for $h = 0$ is significantly distinct from the vanishing (17). From the natural assignment $\frac{4\pi}{h}\sin(\pi h(xy' - yx')) = 4\pi^2(xy' - yx')$ for $h = 0$ we get the Poisson brackets for the Fourier transforms of k' and k defined on $\mathcal{O}_0$ (13):

$$\rho_{(q,p)}\{[k', k]\} = \frac{\partial \hat{k}'}{\partial q}\frac{\partial \hat{k}}{\partial p} - \frac{\partial \hat{k}'}{\partial p}\frac{\partial \hat{k}}{\partial q}. \tag{21}$$

Furthermore the dynamical equation based on the modified commutator (19) with a suitable Hamilton type function $H(s, x, y)$ for an observable $f(s, x, y)$ on $\mathbb{H}^n$

$$\dot{f} = \{[H, f]\} \text{ is reduced } \begin{cases} \text{by } \rho_h,\ h \neq 0 \text{ on } \mathcal{O}_h \text{ (6) to Moyal's equation [30, (8)];} \\ \text{by } \rho_{(q,p)} \text{ on } \mathcal{O}_0 \text{ (13) to Poisson's equation [1, § 39].} \end{cases} \tag{22}$$

The same connections are true for the solutions of these three equations, see [26] for the harmonic oscillator and [4, 5] for forced oscillator examples.

3 De Donder–Weyl field theory

We extend p-mechanics to the De Donder–Weyl field theory, see [12–20] for detailed exposition and further references. We will be limited here to the preliminary discussion which extends the comment 5.2.(1) from the earlier paper [25]. Our notations will slightly different from the used in the papers [12–20] to make it consistent with the used above and avoid clashes.

3.1 Hamiltonian form of field equation

Let the underlying space-time have dimension and $n+1$ parametrised by coordinates u^μ, $\mu = 0, 1, \ldots, n$ (with u^0 parameter traditionally associated with a time-like direction). Let a field be described by m component tensor q^a, $a = 1, \ldots, m$. For a system defined by a Lagrangian density $L(q^a, \partial_\mu q^a, u^\mu)$ De Donder–Weyl theory suggests new set of *polymomenta* p_a^μ and *DW Hamiltonian function* $H(q^a, p_a^\mu, u^\mu)$ defined as follows:

$$p_a^\mu := \frac{\partial L(q^a, \partial_\mu q^a, u^\mu)}{\partial(\partial_\mu y^a)} \qquad \text{and} \qquad H(q^a, p_a^\mu, u^\mu) = p_a^\mu\, \partial_\mu q^a - L(q^a, \partial_\mu q^a, u^\mu). \tag{23}$$

Consequently the Euler–Lagrange field equations could be transformed to the Hamilton form:

$$\frac{\partial q^a}{\partial u^\mu} = \frac{\partial H}{\partial p_a^\mu}, \qquad \frac{\partial p_a^\mu}{\partial u^\mu} = -\frac{\partial H}{\partial q^a}, \tag{24}$$

with the standard summation (over repeating indexes) agreement. The main distinction from a particle mechanics is the existence of $n+1$ different polymomenta p_a^μ associated to each field variable q^a. Correspondingly particle mechanics could be considered as a particular case when $n+1$ dimensional space-time degenerates for $n=0$ to "time only".

The next two natural steps [12–20] inspired by particle mechanics are:

1. Introduce an appropriate Poisson structure, such that the Hamilton equations (24) will represent the Poisson brackets.
2. Quantise the above Poisson structure by some means, e.g. Dirac–Heisenberg–Shrödinger–Weyl technique or geometric quantisation.

We use here another path: first to construct a p-mechanical model for equations (24) and then deduce its quantum and classical derivatives as was done for the particle mechanics above. To simplify presentation we will start from the scalar field, i.e. $m=1$. Thus we drop index a in q^a and p_a^μ and simply write q and p^μ instead.

We also assume that underlying space-time is flat with a constant metric tensor $\eta^{\mu\nu}$. This metric define a related *Clifford algebra* [3, 6, 8] with generators e^μ satisfying the relations

$$e^\mu e^\nu + e^\nu e^\mu = \eta^{\mu\nu}. \tag{25}$$

Remark 1. For the Minkowski space-time (i.e. in the context of special relativity) a preferable choice is *quaternions* [29] with generators i, j, k instead the general Clifford algebra.

Since q and p^μ look like conjugated variables p-mechanics suggests that they should generate a Lie algebra with relations similar to (4). The first natural assumption is the $n+3(=1+(n+1)+1)$-dimensional Lie algebra spanned by X, Y_μ, and S with the only non-trivial commutators $[X, Y_\mu] = S$. However as follows from the Kirillov theory [21] any its unitary irreducible representation is limited to a representation of $\mathbb{H}^1$ listed by the Stone–von Neumann Theorem 1. Consequently there is little chances that we could obtain the field equations (24) in this way.

3.2 *p*-mechanical approach to the field theory

The next natural candidate is the Galilean group $\mathbb{G}^{n+1}$, i.e. a nilpotent step 2 Lie group with the $2n+3(=1+(n+1)+(n+1))$-dimensional Lie algebra. It has a basis X, Y_μ, and S_μ with $n+1$-dimensional centre spanned by S_μ. The only non-trivial commutators are

$$[X, Y_\mu] = S_\mu, \qquad \text{where} \quad \mu = 0, 1, \ldots, n. \tag{26}$$

Again the Kirillov theory [21] assures that any its *complex valued* irreducible representation is a representation of $\mathbb{H}^1$, but multidimensionality of the centre offers an option [7, 24] to consider *Clifford valued* representations of $\mathbb{G}^{n+1}$. Thus we proceed with this group.

Remark 2. The appearance of Clifford algebra in connection with field theory and space-time geometry is natural. For example, the conformal invariance of space-time has profound consequences in astrophysics [28] and, in their turn, conformal (Möbius) transformations are most naturally represented by linear-fractional transformations in Clifford algebras [6]. Some other links between nilpotent Lie groups and Clifford algebras are listed in [24].

The Lie group $\mathbb{G}^{n+1}$ as manifold homeomorphic to $\mathbb{R}^{2n+3}$ with coordinates (s,x,y), where $x \in \mathbb{R}$ and $s, y \in \mathbb{R}^{n+1}$. The group multiplication in this coordinates defined by, cf. (1):

$$\begin{aligned}(s,x,y)*(s',x',y') \\ = \left(s_0+s_0'+\frac{1}{2}\omega(x,y_0;x',y_0'),\ldots,s_n+s_n'+\frac{1}{2}\omega(x,y_n;x',y_n'),x+x',y+y'\right).\end{aligned} \tag{27}$$

Observables are again defined as convolution operators on $L_2(\mathbb{G}^{n+1})$. To define an appropriate brackets of two observables k_1 and k_2 we will again modify their commutator $[k_1,k_2]$ by antiderivative operators $\mathcal{A}_0$, $\mathcal{A}_1$, …, $\mathcal{A}_n$ which are multiples of right inverse to the vector fields S_0, S_1, …, S_n, cf. (18):

$$S_\mu \mathcal{A}_\mu = 4\pi^2 I, \quad \text{where} \quad \mathcal{A}_\mu e^{2\pi i h s_\mu} = \begin{cases} \frac{2\pi}{ih} e^{2\pi i h s_\mu}, & \text{if } h \neq 0, \\ 4\pi^2 s_\mu, & \text{if } h = 0, \end{cases} \quad \text{and} \quad \mu = 0,1,\ldots,n. \tag{28}$$

The definition of the brackets follows the ideas of [7, § 3.3]: to each vector field S_μ should be associated a generator e^μ of Clifford algebra (25). Thus our brackets are as follows, cf. (19):

$$\{[B_1,B_2]\} = B_1 * \mathcal{A}B_2 - B_2 * \mathcal{A}B_1, \qquad \text{where} \quad \mathcal{A} = e^\mu \mathcal{A}_\mu. \tag{29}$$

These brackets will be used in the right-hand side of the p-dynamic equation. Its left-hand side should contain a replacement for the time derivative. As was already mentioned in [12–20] the space-time play a rôle of multidimensional time in the De Donder–Weyl construction. Thus we replace time derivative by the symmetric pairing $D\circ$ with the *Dirac operator* [3, 6, 8] $D = e^\mu \partial_\mu$ as follows:

$$D \circ f = -\frac{1}{2}\left(e^\mu \frac{\partial f}{\partial u^\mu} + \frac{\partial f}{\partial u^\mu} e^\mu\right), \qquad \text{where} \ \ D = e^\mu \partial_\mu. \tag{30}$$

Finally the p-mechanical dynamic equation, cf. (22):

$$D \circ f = \{[H,f]\}, \tag{31}$$

is defined through the brackets (29) and the Dirac operator (30).

To "verify" the equation (31) we will find its classical representation and compare it with (24). Similarly to calculations in section 2.2 we find, cf. (21):

$$\rho_{(q,p^\mu)}\left\{[k',k]\right\} = \frac{\partial \hat{k}'}{\partial q} e^\mu \frac{\partial \hat{k}}{\partial p^\mu} - \frac{\partial \hat{k}'}{\partial p^\mu} e^\mu \frac{\partial \hat{k}}{\partial q}. \tag{32}$$

Consequently the dynamic of field observable q from the equation (31) with a scalar-valued Hamiltonian H is given by:

$$D \circ q = \left(\frac{\partial H}{\partial q} e^\mu \frac{\partial}{\partial p^\mu} - \frac{\partial H}{\partial p^\mu} e^\mu \frac{\partial}{\partial q}\right) q \qquad \Longleftrightarrow \qquad \frac{\partial q}{\partial u^\mu} e^\mu = \frac{\partial H}{\partial p^\mu} e^\mu, \tag{33}$$

i.e. after separation of components with different generators e^μ we get first $n+1$ equations from (24).

To get the last equation for polymomenta (24) we again use the Clifford algebra generators to construct the *combined polymomenta* $p = e_\nu p^\nu$. For them:

$$\begin{aligned} D \circ p &= -\frac{1}{2}\left(e^\mu \frac{\partial e_\nu p^\nu}{\partial u^\mu} + \frac{\partial e_\nu p^\nu}{\partial u^\mu} e^\mu\right) = -\frac{\partial p^\mu}{\partial u^\mu} e^\mu e_\mu, \\ \{[H,f]\} &= \frac{\partial H}{\partial q} e^\mu \frac{\partial e_\nu p^\nu}{\partial p^\mu} - \frac{\partial H}{\partial p^\mu} e^\mu \frac{\partial e_\nu p^\nu}{\partial q} = \frac{\partial H}{\partial q} e^\mu e_\mu. \end{aligned}$$

Thus the equation (31) for the combined polymomenta $p = e_\nu p^\nu$ becomes:

$$\frac{\partial p^\mu}{\partial u^\mu} e^\mu e_\mu = -\frac{\partial H}{\partial q} e^\mu e_\mu, \tag{34}$$

i.e. coincides with the last equation in (24) up to a constant factor $e^\mu e_\mu$.

Consequently images of the equation (31) under the infinite dimensional representation of the group $\mathbb{G}^{n+1}$ could stand for quantisations of its classical images in (24), (32). A further study of quantum images of the equation (31) as well as extension to vector or spinor fields should follow in subsequent papers [27].

Remark 3. To consider vector or spinor fields with components q_a, $a = 1, \ldots, m$ it worths to introduce another Clifford algebra with generators c^a and consider a composite field $q = c^a q_a$. There are different ways to link Clifford and Grassmann algebras, see e.g. [2, 10]. Through such a link the Clifford algebra with generators e_μ corresponds to horizontal differential forms in the sense of [12–20] and the Clifford algebra generated by c_a – to the vertical.

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