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$SU(N)$ Flavor Dynamics within a Generalized Heat Kernel Expansion

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The asymptotic expansion of the one-loop effective action $W = \ln\det A$ is derived for the case in which a non-degenerate mass matrix $\mathcal{M} = \mathrm{diag}(m_1, m_2, \ldots)$ is considered, generalizing the standard method of Schwinger–DeWitt. The positively defined elliptic operator $A = U + \mathcal{M}^2$ depends on the external classical fields taking values in the Lie algebra of the internal symmetry group G. The first coefficients of the new asymptotic series are calculated and their relationship with the Seeley–DeWitt coefficients is clarified.

1 Introduction

The existence of fermion families of non-degenerate masses in the Standard Model calls for a field-theoretical framework capable of incorporating in a chiral and gauge covariant way the occurrence of different masses in loops. In QCD, underlying the non-degenerate mass matrix of heavy (constituent) quarks, which results from spontaneous breakdown of chiral symmetry, is a non-degenerate current quark matrix, which dictates the explicit symmetry breaking pattern of the Lagrangian. Both, the covariant and the explicit symmetry breaking terms of the Lagrangian must be preserved in a sensible calculational scheme. Furthermore, in the low-energy regime of QCD, it is more natural to transform to mesonic/baryonic degrees of freedom, which can be obtained by standard bosonization techniques. In this case the bosonization must of course comply with all symmetry requirements of the original Lagrangian.

These issues have been dealt with in various ways, however the symmetries are not automatically fulfilled. For instance [1], through the evaluation of Feynman amplitudes, the symmetry requirements of the original Langrangian have to be checked aposteriori case for case. Perturbative approaches, where the difference in masses is considered to be the small expansion parameter [2], must also be subject to a careful control of symmetries.

In the following we present a technique, which we developed to handle non-degenerate mass matrices, fulfilling by construction all the symmetries of the Lagrangian. It is based on the method of Schwinger–DeWitt [3,4], which is an extremely powerful tool when explicit covariance of calculations of radiative corrections is needed at all intermediate steps. This is the case, for example, for gauge theory [4], quantum gravity [5], chiral field theory [6]; for a recent review see [7].

The main object of study in this formalism is the determinant of the positively defined elliptic operator, which describes quadratic fluctuations of quantum fields in presence of some background fields and contains in compact form the whole information about the one-loop contribution of quantum fields. The proper time formalism of Schwinger allows to evaluate it. The result is an asymptotic expansion of the effective action in powers of proper time with Seeley–DeWitt coefficients a_n [4, 8], which accumulate the whole dependence on background fields. A remarkable property is that every term of the expansion is invariant with respect to transformations of the internal symmetry group. This is a consequence of the general covariance inherent to the formalism. At the present time the asymptotic coefficients a_n are well known up to and including $n = 5$ for a general operator of Laplace type. Details can be found, for instance, in [6, 9].

In the case of massive quantum fields with a degenerate mass matrix $\mathcal{M} = \mathrm{diag}(m, m, \ldots)$, it is not difficult to derive from the proper time expansion an expansion in inverse powers of m^2, since the mass dependence is easily factorized and a subsequent integration over proper time leads to the desired result. The resulting asymptotic coefficients remain the same. If the mass matrix is not degenerate, i.e. $\mathcal{M} = \mathrm{diag}(m_1, m_2, \ldots)$, than its total factorization is impossible because of the noncommutativity of $\mathcal{M}$ with the rest of the elliptic operator. At the same time a naive factorization by parts breaks the covariant character of the asymptotic series. The natural question arises: is there any simple way to follow which leads to factorization, conserving at the same time the explicit covariance of the expansion? Fortunately, there is such a method [10, 11]. It leads to the generalization of the Seeley–DeWitt coefficients. Here we will not consider the manifest chiral symmetry breaking effect. A careful analysis of this problem would lead us too far away from the subject, leaving the present result for the covariantly transforming terms without changes. We refer to [11] where the symmetry breaking terms have been considered by breaking the $SU(2) \times SU(2)$ chiral symmetry of the Nambu–Jona–Lasinio model [12] by the current quark mass matrix with $\hat{m}_u \neq \hat{m}_d$.

2 Integral representation for the one-loop contribution

The logarithm of a formal determinant describes the low order radiative corrections to the classical theory. Let fermion fields play the role of virtual quanta producing those corrections and the scalar and pseudoscalar mesons be external background fields. In this case the real part of the corresponding effective action can be represented as a proper time integral

$$W[Y] = -\ln|\det D| = \frac{1}{2}\int_0^\infty \frac{dt}{t}\rho(t, \Lambda^2)\mathrm{Tr}\left(e^{-tD^\dagger D}\right). \qquad (1)$$

Here the Dirac operator D depends on the background fields, which are collected in the Hermitian second-order differential elliptic operator $D^\dagger D = \mathcal{M}^2 + B = -\partial^2 + Y + \mathcal{M}^2$ in the term Y. We use the Euclidian metric to define the effective action $W[Y]$. To make the integral over t convergent at the lower limit, the regulator $\rho(t, \Lambda^2)$ has been added in (1). All our conclusions are independent of the form of this function, which includes the ultraviolet cut off Λ, in the sense that the generalized Seeley–DeWitt coefficients do not depend on the regulator.

Let us assume that we are dealing with chiral field theory, which possesses the global $U(N_f)_L \times U(N_f)_R$ symmetry if the fermion fields are massless. The typical example is quantum chromodynamics, excluding the $U_A(1)$ anomaly. It is known that the vacuum state of low energy QCD is noninvariant with respect to the action of the chiral group and the whole system makes a phase transition to the state with massive quarks. If one takes into account that the explicit chiral symmetry breaking takes place in QCD through the mass terms of current quarks one can conclude that not equal current quark masses will lead also to not equal constituent quark masses. In order to study this system at large distances we will need the expansion of the effective action in inverse powers of the non degenerate mass matrix

$$\mathcal{M} = \sum_{i=1}^{N_f}\mathcal{M}_i E_i, \qquad (E_i)_{jk} = \delta_{ij}\delta_{ik}, \qquad E_i E_j = \delta_{ij}E_j. \qquad (2)$$

The orthonormal basis E_i belongs to the flavor space where the chiral group acts on quarks and background fields in accordance with their transformation properties.

The heat kernel $\mathrm{Tr}\,[\exp(-tD^\dagger D)]$ can be represented [3] as a matrix element of an operator acting in the abstract (unphysical) Hilbert space:

$$\mathrm{Tr}\left(e^{-tD^\dagger D}\right) = \int d^4x\,\mathrm{tr}\,\langle x|e^{-tD^\dagger D}|x\rangle. \tag{3}$$

The plane wave basis $|p\rangle$ significantly simplifies our calculations and leads to the integral representation

$$W[Y] = \frac{1}{2}\int d^4x \int \frac{d^4p}{(2\pi)^4}\int_0^\infty \frac{dt}{t^3}\rho(t,\Lambda^2)e^{-p^2}\mathrm{tr}\left(e^{-t(\mathcal{M}^2+A)}\right)\cdot 1. \tag{4}$$

Here $A = B - 2ip\partial/\sqrt{t}$, and the trace is calculated in flavor space.

3 Asymptotic Schwinger–DeWitt expansion

Before proceeding with our calculations this is the right place to say several words about the standard asymptotic Schwinger–DeWitt expansion. If the mass matrix $\mathcal{M}$ has the degenerate form, than $[\mathcal{M}, A] = 0$ and we find

$$\mathrm{tr}\left(e^{-t(\mathcal{M}^2+A)}\right) = e^{-tm^2}\mathrm{tr}\left(e^{-tA}\right) = e^{-tm^2}\mathrm{tr}\left(\sum_{n=0}^{\infty} t^n a_n\right). \tag{5}$$

Here a_n are the Seeley–DeWitt coefficients which depend on background fields and their derivatives. The integration over momentum and proper time in (4) can be readily done and we obtain the well known result

$$W[Y] = \int \frac{d^4x}{32\pi^2}\sum_{n=0}^{\infty} J_{n-1}\left(m^2\right)\,\mathrm{tr}\,(a_n), \tag{6}$$

where integrals $J_n(m^2)$ are given by

$$J_n(m^2) = \int_0^\infty \frac{dt}{t^{2-n}}e^{-tm^2}\rho\left(t,\Lambda^2\right). \tag{7}$$

Independently on the type of regularization the following property is fulfilled

$$J_n(m^2) = \left(-\frac{\partial}{\partial m^2}\right)^n J_0\left(m^2\right). \tag{8}$$

Choosing $\rho(t,\Lambda^2) = 1 - (1 + t\Lambda^2)e^{-t\Lambda^2}$, which corresponds to two subtractions, one finds

$$J_0(m^2) = \Lambda^2 - m^2\ln\left(1 + \frac{\Lambda^2}{m^2}\right). \tag{9}$$

We see that functions $J_n(m^2)$, starting from $n > 1$, have the asymptotic behavior $J_n(m^2) \sim m^{-2(n-1)}$ at large values of m^2, i.e. we obtain the inverse mass expansion. To warrant the convergence of the series it is necessary not only that the mass m be different from zero, but also that the background fields change slowly at distances of the order of the Compton wavelength $(1/m)$ of the fermion field. If these criteria are not fulfilled then the creation of real pairs gets essential and this expansion is not useful for calculations.

The remarkable property of the considered expansion is the gauge covariance of the Seeley–DeWitt coefficients.

4 Non-degenerate mass case

Let us return back to the formula (4) and show the way how to develop the above-mentioned tool for the case $[\mathcal{M}, A] \neq 0$. As a first step let us use the formula

$$\mathrm{tr}\left(e^{-t(\mathcal{M}^2+A)}\right) = \mathrm{tr}\left(e^{-t\mathcal{M}^2}\left[1+\sum_{n=1}^{\infty}(-1)^n f_n(t,A)\right]\right), \tag{10}$$

to factorize an exponent with the non commuting mass matrix $\mathcal{M}$. Here

$$f_n(t,A) = \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n A(s_1)A(s_2)\cdots A(s_n), \tag{11}$$

where $A(s) = e^{s\mathcal{M}^2} A e^{-s\mathcal{M}^2}$. It is true, that

$$\mathrm{tr}\left[e^{-t\mathcal{M}^2} f_n(t,A)\right] = \frac{t^n}{n!} \sum_{i_1,i_2,\ldots i_n}^{N_f} c_{i_1 i_2 \ldots i_n}(t) \frac{1}{n} \sum_{perm} \mathrm{tr}\left(A_{i_1} A_{i_2} \ldots A_{i_n}\right), \tag{12}$$

where $A_i = E_i A$ by definition, a second summation means that n possible cyclic permutations of operators inside the trace must be done. The coefficients $c_{i_1 i_2 \ldots i_n}(t)$ are totally symmetric with respect to permutations of indices and are calculated easily [11], for instance

$$\begin{aligned}
c_i(t) &= e^{-tm_i^2}, \qquad c_{ij}(t) = \frac{e^{-tm_i^2} - e^{-tm_j^2}}{t\Delta_{ji}}, \\
c_{ijk}(t) &= \frac{2}{t^2}\left(\frac{e^{-tm_i^2}}{\Delta_{ji}\Delta_{ki}} + \frac{e^{-tm_j^2}}{\Delta_{kj}\Delta_{ij}} + \frac{e^{-tm_k^2}}{\Delta_{ik}\Delta_{jk}}\right), \\
c_{ijkl} &= \frac{3!}{t^3}\left(\frac{e^{-tm_i^2}}{\Delta_{li}\Delta_{ki}\Delta_{ji}} + \frac{e^{-tm_j^2}}{\Delta_{ij}\Delta_{lj}\Delta_{kj}} + \frac{e^{-tm_k^2}}{\Delta_{jk}\Delta_{ik}\Delta_{lk}} + \frac{e^{-tm_l^2}}{\Delta_{kl}\Delta_{jl}\Delta_{il}}\right).
\end{aligned} \tag{13}$$

Here $\Delta_{ij} \equiv m_i^2 - m_j^2$. In the case of coincidence of indices one can get that $c_i = c_{ii} = c_{ii\ldots i}$.

Therefore the heat kernel is represented as a sum, every term of which contains the coefficient $c_{ij\ldots}(t)$, multiplyed by the corresponding trace from the product of operators A_i. The following integration over t replaces the t-dependent part in these terms by the integrals $J_l(m_i^2)$ and we finally will face the following problem: every term of this expansion will not be invariant with respect to the transformations of the chiral group, although the total effective action $W[Y]$ will possess this property. One needs the algorithm which automatically groups the terms in chiral invariant blocks. This problem has been solved in the papers [11] on the basis of recurrence relations

$$J_l\left(m_j^2\right) - J_l\left(m_i^2\right) = \sum_{n=1}^{\infty} \frac{\Delta_{ij}^n}{2^n n!}\left[J_{l+n}\left(m_i^2\right) - (-1)^n J_{l+n}\left(m_j^2\right)\right]. \tag{14}$$

There it has been shown that the essence of the problem is confined to the correct choosing of the factorized combination, built from the functions $J_i(m_j^2)$, namely, the effective action $W[Y]$ must be represented in the form

$$W[Y] = \int \frac{d^4x}{32\pi^2} \sum_{i=0}^{\infty} I_{i-1}\, \mathrm{tr}\left(b_i\right), \qquad I_i \equiv \frac{1}{N_f}\sum_{j=1}^{N_f} J_i\left(m_j^2\right). \tag{15}$$

In this case the background fields are *automatically* combined in the covariant coefficients b_i. For example, for $N_f = 3$ the first four coefficients are

$$
\begin{aligned}
& b_0 = 1, \qquad b_1 = -Y, \qquad b_2 = \frac{Y^2}{2} + \frac{\Delta_{12}}{2}\lambda_3 Y + \frac{1}{2\sqrt{3}}\left(\Delta_{13} + \Delta_{23}\right)\lambda_8 Y, \\
& b_3 = -\frac{Y^3}{3!} - \frac{1}{12}(\partial Y)^2 - \frac{1}{12}\Delta_{12}\left(\Delta_{31} + \Delta_{32}\right)\lambda_3 Y \\
& \quad + \frac{1}{12\sqrt{3}}\left[\Delta_{13}(\Delta_{21} + \Delta_{23}) + \Delta_{23}(\Delta_{12} + \Delta_{13})\right]\lambda_8 Y \\
& \quad + \frac{1}{4\sqrt{3}}\left(\Delta_{31} + \Delta_{32}\right)\lambda_8 Y^2 + \frac{1}{4}\Delta_{21}\lambda_3 Y^2.
\end{aligned} \tag{16}
$$

Several comments are in order here. First of all it is obvious that in the limiting case of equal masses $m_1 = m_2 = \cdots = m_{N_f}$ our result coincides with the standard Schwinger–DeWitt expansion. If the masses are not equal, the series (15) is a generalization of the well-known result (6). Instead of the asymptotic Seeley–DeWitt coefficients a_n come the coefficients b_n. Indeed one can check that if the operator $D^\dagger D$ transforms in the adjoint representation $\delta(D^\dagger D) = i[\omega, D^\dagger D]$, then b_n are also covariant, i.e. $\delta b_n = i[\omega, b_n]$, where $\omega = \alpha + \gamma_5 \beta$ are the parameters of the global infinitesimal chiral transformations. It also should be mentioned that, rigorously speaking, the obtained expansion is not an exact inverse mass expansion. Although the integrals I_l for $l \geq 1$ possess the necessary form for asymptotic behavior $I_{l+1}(m_i^2) \sim m_i^{-2l}$, the coefficients b_l depend on the differences of masses, which may influence the character of the expansion. However general symmetry requirements are a more serious and stringent argument in favor of the obtained series as compared to the result of the thorough study [2] based only on the idea of a $1/m^2$ expansion.

5 Relation between coefficients b_n and a_n

The problem of the calculation of the generalized heat kernel coefficients is a more complicated mathematical problem than the calculation of the standard Seeley–DeWitt coefficients. However one can significantly simplify this problem [13], if from the very beginning one uses the transformation properties of the coefficients b_n. Indeed, let us return back to the formula (5) and extend it to the case $[\mathcal{M}, Y] \neq 0$, omitting for simplicity all terms with derivatives in A. It is remarkable that already at this stage one can take into account the two main conclusions which we have found in the previous section: the form of the factorized part depending on the mass and gauge covariance of coefficients b_n. The first aim can be reached through the definition

$$
\langle A \rangle \equiv \frac{\mathrm{tr}\,(A)}{\mathrm{tr}\,(1)}, \qquad \bar{\mathcal{M}}^2 = \mathcal{M}^2 - \langle \mathcal{M}^2 \rangle, \tag{17}
$$

the second one through putting

$$
\left(\bar{\mathcal{M}}^2 + Y\right)^n \equiv \sum_{k=0}^{n} (-1)^k C_n^k \langle \left(\bar{\mathcal{M}}^2\right)^{n-k} \rangle b_k. \tag{18}
$$

Since the left side of equation (18) transforms covariantly, i.e. $\left(\bar{\mathcal{M}}^2 + Y\right)^\Omega = \Omega^{-1}\left(\bar{\mathcal{M}}^2 + Y\right)\Omega$, and on the right side the term $\langle \bar{\mathcal{M}}^2 \rangle$ is invariant with respect to the chiral transformations Ω, it is obvious, that $b_k(Y^\Omega, \bar{\mathcal{M}}^{2\Omega}) = \Omega^{-1} b_k\left(Y, \bar{\mathcal{M}}^2\right)\Omega$. To see how these definitions work let us consider the exponent in the heat kernel (5)

$$\begin{aligned} e^{-t(\mathcal{M}^2+Y)} &= e^{-t\langle\mathcal{M}^2\rangle}\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\left(\bar{\mathcal{M}}^2+Y\right)^n \\ &= e^{-t\langle\mathcal{M}^2\rangle}\langle e^{-t\bar{\mathcal{M}}^2}\rangle\sum_{n=0}^{\infty}\frac{t^n}{n!}b_n = \langle e^{-t\mathcal{M}^2}\rangle\sum_{n=0}^{\infty}\frac{t^n}{n!}b_n. \end{aligned} \tag{19}$$

Since

$$\langle e^{-t\mathcal{M}^2}\rangle = \sum_{i=1}^{N_f} e^{-tm_i^2}\langle E_i\rangle = \frac{1}{N_f}\sum_{i=1}^{N_f} e^{-tm_i^2}, \tag{20}$$

the following integration over t leads us to the known result (15), and the coefficients b_n can be found using the definition (18). The same definition allows us to relate b_n with the linear combination of the standard Seeley–DeWitt coefficients $a_i(Y)$, where the replacement $Y \to Y + \bar{\mathcal{M}}^2$ should be done [13]

$$b_n = \sum_{i=0}^{n}\frac{\beta_{n-i}}{(n-i)!}a_i\big(Y \to Y + \bar{\mathcal{M}}^2\big). \tag{21}$$

The parameters β_i depend only on the masses of fermion fields. To establish the form of this dependence it is sufficient to calculate b_n in the simplest case, when in the elliptic operator under consideration all terms with derivatives are omitted. This problem is much simpler to solve than the calculation of the coefficients b_n from the very beginning.

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