Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 3, 1294–1300

On Geodesical Extension of the Schwarzshild Space-Time

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The properties of geodesical extension of the Schwarzshild metric for the radial motion of test particle are studied.

1 Introduction

The notion of the Riemann extension of non-Riemannian spaces was introduced first in [1]. Main idea of this theory is to apply the methods of Riemann geometry for studying of the properties of non-Riemannian spaces.

For example, the system of differential equations in form

$$\frac{d^2x^k}{ds^2} + \Pi^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{1}$$

with arbitrary coefficients $\Pi^k_{ij}(x^l)$ can be considered as the system of geodesic equations of affinely connected space with local coordinates x^k.

For the n-dimensional Riemannian spaces with the metrics

$$^n ds^2 = g_{ij}dx^i dx^j$$

the system of geodesic equations looks similar, but the coefficients $\Pi^k_{ij}(x^l)$ now have very special form and depend on the choice of the metric g_{ij}

$$\Pi^i_{kl} = \Gamma^i_{kl} = \frac{1}{2}g^{im}(g_{mk,l} + g_{ml,k} - g_{kl,m}).$$

In order for methods of Riemann geometry to be applied for studying of the properties of the spaces with equations (1) construction of $2n$-dimensional extension of the space with local coordinates x^i was introduced.

The metric of extended space constructs with help of coefficients of equation (1) and looks as follows

$$^{2n}ds^2 = -2\Pi^k_{ij}(x^l)\Psi_k dx^i dx^j + 2d\Psi_k dx^k, \tag{2}$$

where Ψ_k are the coordinates of additional space.

The important property of such metric is that the geodesic equations of metric (2) consist of two parts

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0, \tag{3}$$

and

$$\frac{\delta^2\Psi_k}{ds^2} + R^l_{kji}\dot{x}^j\dot{x}^i\Psi_l = 0, \tag{4}$$

where

$$\frac{\delta\Psi_k}{ds} = \frac{d\Psi_k}{ds} - \Gamma^l_{jk}\Psi_l\frac{dx^j}{ds}.$$

The first part (3) of the complete system is the system of equations for geodesics of basic space with local coordinates x^i and it does not contains the coordinates Ψ_k.

The second part (4) of system of geodesic equations has the form of linear 4×4 matrix system of second order ODE's for coordinates Ψ_k

$$\frac{d^2\vec{\Psi}}{ds^2} + A(s)\frac{d\vec{\Psi}}{ds} + B(s)\vec{\Psi} = 0 \tag{5}$$

with the matrix

$$A = A(x^i(s), \dot{x}^i(s)), \qquad B = B(x^i(s), \dot{x}^i(s)).$$

From this point of view we have the case of geodesical extension of the basic space (x^i).

It is important to note that the geometry of extended space is connected with geometry of basic space.

For example the property of such space to be a Ricci-flat one keeps also for the extended space.

This fact give us the possibility to use the linear system of equation (5) for studying of the properties of basic space.

In particular the invariants of the 4×4 matrix-function

$$E = B - \frac{1}{2}\frac{dA}{ds} - \frac{1}{4}A^2$$

under change of the coordinates Ψ_k can be used for that.

The first application of the notion of extended spaces to study of nonlinear second-order differential equations connected with nonlinear dynamical systems was made in paper of the author [2–4].

Here we consider the properties of extended spaces for the Einstein-spaces in General Relativity.

2 The Schwarzschild space-time and geodesic equations

The line element of standard metric of the Schwarzschild space-time in coordinate system x, θ, ϕ, t has the form

$$ds^2 = \frac{1}{(1-2M/x)}dx^2 + x^2(d\theta^2 + \sin^2\theta d\phi^2) - (1-2M/x)dt^2. \tag{6}$$

The geodesic equations of this type of the metric are given by

$$\ddot{x} + \frac{M}{x(2M-x)}\dot{x}^2 + (2M-x)\dot{\theta}^2 + (2M-x)\sin^2\theta\dot{\phi}^2 - \frac{M(2M-x)}{x^3}\dot{t}^2 = 0, \tag{7}$$

$$\ddot{\theta} + \frac{2}{x}\dot{x}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{8}$$

$$\ddot{\phi} + \frac{2}{x}\dot{x}\dot{\phi} + 2\frac{\cos\theta}{\sin\theta}\dot{\phi}\dot{\theta} = 0, \tag{9}$$

$$\ddot{t} - 2\frac{M}{x(2M-x)}\dot{x}\dot{t} = 0. \tag{10}$$

The symbols of Christoffel of the metric (6) look as

$$\Gamma^1_{11} = \frac{M}{x(2M-x)}, \qquad \Gamma^1_{22} = (2M-x), \qquad \Gamma^1_{33} = (2M-x)\sin^2\theta,$$
$$\Gamma^1_{44} = -\frac{M(2M-x)}{x^3}, \qquad \Gamma^2_{12} = \frac{1}{x}, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta, \qquad \Gamma^3_{23} = \frac{\cos\theta}{\sin\theta},$$
$$\Gamma^4_{14} = -\frac{M}{x(2M-x)}, \qquad \Gamma^3_{13} = \frac{1}{x}.$$

The equations of geodesic (7)–(10) have the first integrals

$$x^4\dot{x}^2 = E^2x^4 - (x^2-2Mx)(\mu^2x^2+L^2), \quad x^4\dot{\theta}^2 = L^2 - \frac{L^2}{\sin^2\theta}, \tag{11}$$
$$\dot{\phi} = \frac{L}{x^2\sin^2\theta}, \qquad \dot{t} = \frac{E}{(1-2M/x)}. \tag{12}$$

where a dot denotes differentiation with respect to parameter s and (E,μ,L) are the constants of motion.

3 The Riemann extension of the Schwarzschild metric

Now with the help of the formulae (2) we construct the eight-dimensional extension of basic metric (6)

$$\begin{aligned} ds^2 = &-\frac{2M}{x(2M-x)}Pdx^2 - \frac{2}{x}Qdxd\theta - 2(2M-x)Pd\theta^2 - \frac{2}{x}Udxd\phi + 2\frac{M}{x(2M-x)}Vdxdt \\ &- 2\frac{\cos\theta}{\sin\theta}Ud\phi d\theta - 2((2M-x)\sin^2\theta P - \sin\theta\cos\theta Q)d\phi^2 + 2\frac{M(2M-x)}{x^3}Pdt^2 \\ &+ 2dxdP + 2d\theta dQ + 2d\phi dU + 2dtdV, \end{aligned} \tag{13}$$

where (P,Q,U,V) are the set of additional coordinates.

The eight-dimensional space in local coordinates $(x,\theta,\phi,t,P,Q,U,V)$ with this type of metric is also the Einstein space with the condition on the Ricci tensor

$$^8R_{ik} = 0.$$

The complete system of geodesic equations for the metric (7) is divided into two parts.

The first part coincides with the equations (7)–(10) on the coordinates (x,θ,ϕ,t) and second part forms the linear system of equations for coordinates P, Q, U, V.

They are defined as

$$\begin{aligned} &\ddot{P} + \frac{2M}{x(x-2M)}\dot{x}\dot{P} - \frac{2}{x}\dot{\theta}\dot{Q} - \frac{2}{x}\dot{\phi}\dot{U} - \frac{2M}{x(x-2M)}\dot{t}\dot{V} \\ &\quad - \left(\frac{2M}{x^2(x-2M)}\dot{x}^2 + \frac{(x-2M)}{x}\dot{\theta}^2 + \frac{\sin^2\theta(x-2M)}{x}\dot{\phi}^2 + \frac{2M(x-2M)}{x^4}\dot{t}^2\right)P \\ &\quad + \left(\frac{4}{x^2}\dot{x}\dot{\theta} - \frac{2\cos\theta}{x}\dot{\phi}^2\right)Q + \left(\frac{4}{x^2}\dot{x}\dot{\phi} + \frac{4\cos\theta}{x\sin\theta}\dot{\theta}\dot{\phi}\right)U + \left(\frac{4M^2}{x^2(x-2m)^2}\dot{x}\dot{t}\right)V = 0, \\ &\ddot{Q} + 2(x-2m)\dot{\theta}\dot{P} - \frac{2}{x}\dot{x}\dot{Q} - \frac{2\cos\theta}{\sin\theta}\dot{\phi}\dot{U} - \frac{2(x-4M)}{x}\dot{x}\dot{\theta}P \\ &\quad + \left(\frac{2(x-3M)}{x^2(x-2M)}\dot{x}^2 - \frac{2(x-2M)}{x}\dot{\theta}^2 - \frac{(x-4M\sin^2\theta)}{x}\dot{\phi}^2 + \frac{2M(x-2M)}{x^4}\dot{t}^2\right)Q \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{4\cos\theta}{x\sin\theta}\dot{x}\dot{\phi}+\frac{4\cos^2\theta}{\sin^2\theta}\dot{\theta}\dot{\phi}\right)U=0,\\
&\ddot{U}+2\sin^2\theta(x-2M)\dot{\phi}\dot{P}+2\sin\theta\cos\theta\dot{\phi}\dot{Q}-\left(\frac{2\cos\theta}{\sin\theta}\dot{\theta}+\frac{2}{x}\dot{x}\right)\dot{U}-\frac{2\sin^2\theta(x-4M)}{x}\dot{x}\dot{\phi}P\\
&\quad-\left(\frac{4\sin\theta\cos\theta}{x}\dot{x}\dot{\phi}+2\dot{\theta}\dot{\phi}\right)Q+\left(\frac{2(x-3M)}{x^2(x-2m)}\dot{x}^2+\frac{4\cos\theta}{x\sin\theta}\dot{x}\dot{\theta}\right.\\
&\quad\left.+\frac{2(x\cos^2\theta+2M\sin^2\theta)}{x\sin^2\theta}\dot{\theta}^2-\frac{2(x-2M\sin^2\theta)}{x}\dot{\phi}^2\right)U=0,\\
&\ddot{V}-\frac{2M(x-2m)}{x^3}\dot{t}\dot{P}-\frac{2M}{x(x-2M)}\dot{x}\dot{V}+\frac{4M(x-2M)}{x^4}\dot{x}\dot{t}P\\
&\quad+\left(\frac{2M(2x-3M)}{x^2(x-2M)^2}\dot{x}^2-\frac{2M}{x}\dot{\theta}^2-\frac{2M\sin^2\theta}{x}\dot{\phi}^2+\frac{2M^2}{x^4}\dot{t}^2\right)V=0.
\end{aligned}
$$

So we get the linear matrix-second order ODE for the coordinates U, V, P, Q

$$
\frac{d^2\Psi}{ds^2}+A(x,\theta,\phi,t)\frac{d\Psi}{ds}+B(x,\theta,\phi,t)\Psi=0, \tag{14}
$$

where

$$
\Psi(s)=\begin{pmatrix}P(s)\\Q(s)\\U(s)\\V(s)\end{pmatrix}
$$

and A, B are some 4×4 matrix-functions depending on the coordinates $x(s)$, $\theta(s)$, $\phi(s)$, $t(s)$ and their derivatives.

We shall study this system of equations at the conditions $\theta=\pi/2$, $L=0$, $M=\mu$ and $\phi(s)=\text{const}$. This is simplest case and it corresponds to the radial motion of the test particle in the Schwarzschild space-time.

With these conditions matrix A takes the form

$$
\begin{bmatrix}
-\frac{2M}{x(-2M+x)}\dot{x} & 0 & 0 & \frac{2M}{x(-2M+x)}\dot{t}\\
0 & \frac{2}{x}\dot{x} & 0 & 0\\
0 & 0 & \frac{2}{x}\dot{x} & 0\\
\frac{2M(-2M+x))}{x^3}\dot{t} & 0 & 0 & \frac{2M}{x(-2M+x)}\dot{x}
\end{bmatrix},
$$

and matrix B has the elements

$$
B_{11}=-\frac{\left(2Mx^2+2Mx^2\dot{x}^2-8M^2x\dot{t}^2+8M^3\dot{t}^2\right)}{x^4(x-2M)},\qquad B_{12}=0,\qquad B_{13}=0,
$$

$$
B_{14}=\frac{4M^2\dot{x}\dot{t}}{x^2(x-2M)^2},\qquad B_{21}=0,
$$

$$
B_{22}=\frac{\left(2x^3\dot{x}^2-8M^2x\dot{t}^2+8M^3\dot{t}^2+2Mx^2\dot{t}^2-6Mx^2\dot{x}^2\right)}{x^4(x-2M)},\qquad B_{23}=0,\qquad B_{24}=0,
$$

$$
B_{31}=0,\qquad B_{32}=0,\qquad B_{33}=2\frac{\left(4M^3\dot{t}^2+x^3\dot{x}^2-3Mx^2\dot{x}^2+Mx\dot{t}^2-4M^2x\dot{t}^2\right)}{x^4(x-2M)},
$$

$$
B_{34}=0,\qquad B_{41}=4\,\frac{M(x-2M)\dot{x}\dot{t}}{x^4},\qquad B_{42}=0,\qquad B_{43}=0,
$$

$$B_{44} = 2M\frac{\left(4M^3\dot{t}^2 + 2x^3\dot{x}^2 - 3Mx^2\dot{x}^2 + Mx^2\dot{t}^2 - 4M^2x\dot{t}^2\right)}{x^4(x-2M)^2}.$$

The equations for the coordinates of x and t are

$$x^4\dot{x}^2 = \mu^2x^4 - (x^2 - 2Mx)(\mu^2x^2) = 2M\mu^2x^2,$$
$$\dot{t} = \frac{\mu}{(1-2M/x)}.$$

They have the solutions

$$x(s) = 1/4\frac{12^{2/3}\left(\mu\, M^2\,(s-C_1)\sqrt{2}\right)^{2/3}}{M} \tag{15}$$

and

$$t(s) = \mu s + 2\frac{\mu\sqrt[3]{3}\sqrt[3]{2}M}{\sqrt[3]{M\mu^2}}\sqrt[3]{s} + \left(-4\,\mu\,M^{3/2}\mathrm{arctanh}\left(\frac{1}{2}\frac{\sqrt[3]{3}\sqrt[3]{2}\sqrt[6]{M\mu^2}}{\sqrt{M}}\sqrt[3]{s}\right) + C_2\sqrt{M\mu^2}\right)\frac{1}{\sqrt{M\mu^2}}, \tag{16}$$

where C_i are parameters.

Now we will solve our matrix system of equations (14) with a given matrix A and B.

The second and third equations of the system are given by

$$\frac{d^2}{ds^2}Q(s) - 2\frac{\dot{x}}{x}\frac{d}{ds}Q(s) - \frac{\left(6Mx^2\dot{x}^2 + 8M^2x\dot{t} - 2x^3\dot{x}^2 - 8M^3\dot{t}^2 - 2Mx^2\dot{t}^2)\right)}{x^4(x-2M)}Q(s) = 0, \tag{17}$$
$$\frac{d^2}{ds^2}U(s) - 2\frac{\dot{x}}{x}\frac{d}{ds}U(s) - \frac{\left(6Mx^2\dot{x}^2 + 8M^2x\dot{t} - 2x^3\dot{x}^2 - 8M^3\dot{t}^2 - 2Mx^2\dot{t}^2)\right)}{x^4(x-2M)}U(s) = 0. \tag{18}$$

The first and fourth equations of the given system have the form of 2×2 matrix system of linear second order differential equations with the variable coefficients

$$\frac{d^2}{ds^2}P(s) + \frac{2M\dot{x}}{x(x-2M)}\frac{d}{ds}P(s) - \frac{2M\dot{t}}{x(x-2M)}\frac{d}{ds}V(s) - \frac{\left(2\,Mx^2\dot{t}^2 + 2Mx^2\dot{x}^2 - 8M^2x\dot{t}^2 + 8M^3\dot{t}^2\right)}{x^4(x-2M)}P(s) + \frac{4Mx^2\dot{x}\dot{t}}{x^2(x-2M)^2}V(s) = 0, \tag{19}$$

and

$$\frac{d^2}{ds^2}V(s) - \frac{2M\,(-2M+x))\,\dot{t}}{x^3}\frac{d}{ds}P(s) - \frac{2M\dot{x}}{x(x-2M)}\frac{d}{ds}V(s) + \frac{4M(x-2M)\dot{x}\dot{t}}{x^4}P(s) + \frac{2M\left(-3Mx^2\dot{x}^2 + 2x^2\dot{x}^2 - 4M^2x\dot{t}^2 + Mx^2\dot{t}^2 + 4M^3\dot{t}^2\right)}{x^4(x-2M)^2}V(s) = 0. \tag{20}$$

To integrate the equation (17) we transform it with the help of substitution

$$Q(s) = x(s)T(s).$$

As a result we get the equation for the function $T(s)$

$$\frac{d^2T}{ds^2} + \frac{\left(x^4\ddot{x} - 2Mx^3\ddot{x} - 2Mx^2\dot{x}^2 - 8M^2x\dot{t}^2 + 8M^3\dot{t}^2 + 2Mx^2\dot{t}^2)\right)}{x^4(x-2M)}T(s) = 0$$

which takes the form

$$\frac{d^2}{ds^2}T(s) + \frac{M\mu^2}{x^3}T(s) = 0 \tag{21}$$

after substitution here the relation (15) and (16).

In the simplest case

$$x(s) = (9M\mu^2/2)^{1/3}s^{2/3}$$

the solution of equation (21) is

$$T(s) = C_1 s^{2/3} + C_2 s^{1/3},$$

where C_i are the parameters.

As a result we get solutions for the coordinates $Q(s)$ and $U(s)$.

They look as

$$Q(s) = (9M\mu^2/2)^{1/3}(C_1 s + C_2 s^{4/3}), \tag{22}$$

$$U(s) = (9M\mu^2/2)^{1/3}(C_3 s + C_4 s^{4/3}). \tag{23}$$

To integrate the equations for the coordinates $P(s)$ and $V(s)$ we use the relation

$$\dot{x}(s)P(s) + \dot{\theta}(s)Q(s) + \dot{\phi}(s)U(s) + \dot{t}(s)V(s) - \frac{s}{2} = \text{const}, \tag{24}$$

which is consequence of the well-known first integral of geodesic equations of arbitrary Riemann space

$$g_{ik}\dot{x}^i\dot{x}^k = \text{const}.$$

In our case the relation (24) takes the form

$$\dot{x}P(s) + \dot{t}V(s) - \frac{s}{2} = 0 \tag{25}$$

(const = 0) and with the help of this condition the system (19), (20) can be reduced into the two independent equations.

In fact, from the equation (25) we get

$$P(s) = -\frac{2\dot{t}V(s) - s}{\dot{x}} \tag{26}$$

and after substitution of this expression into the equation (20) we get one second-order differential equation for the function $V(s)$.

In fact, with the help of relations (11), (12) we have

$$\frac{d^2}{ds^2}V(s) + \frac{\sqrt{2M\mu^2}}{x(s)^{3/2}}\frac{d}{ds}V(s) - \frac{3M\mu^2}{x(s)^3}V(s) - \frac{\sqrt{2M}}{2x(s)^{3/2}} + \frac{3M\mu}{2x(s)^3}s = 0.$$

Now we put here the expression

$$x(s) = (9M\mu^2/2)^{1/3}s^{2/3}$$

for the function x and as a result we get a simple equation

$$\frac{d^2}{ds^2}V(s) + \frac{2}{3s}\frac{d}{ds}V(s) - \frac{2}{3s^2}V(s) = 0$$

for the coordinate $V(s)$.

Its solution is defined by

$$V(s) = C_5 s + \frac{C_6}{s^{2/3}}. \tag{27}$$

Now from the relation (26) we get the expression for coordinate $P(s)$

$$P(s) = -1/2\,\frac{s^{2/3}\left(6\mu\sqrt[3]{\mu^2 M}s^{5/3}C_5 + 6\mu\sqrt[3]{\mu^2 M}C_6 - 3s^{5/3}\sqrt[3]{\mu^2 M} + 2\sqrt[3]{3}\sqrt[3]{2}sM\right)}{\sqrt[3]{\mu^2 M}\left(s\sqrt[3]{\mu^2 M}3^{2/3}2^{2/3} - 4\sqrt[3]{s}M\right)}. \tag{28}$$

So the formulae (22), (23), (27), (28) present the solutions of geodesic equations for coordinates (Q, U, V, P) of Riemann extension of the Schwarzschild metric for the radial motion of test particle in basic space with coordinates (x, θ, ϕ, t).

The generalization and the interpretation of these solutions will be done later.

Acknowledgements

The author thanks INTAS-99-01782 Programme and the Royal Swedish Academy of Sciences for financial support.

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