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On Finiteness of Critical Tits Forms of Posets

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In the paper we consider the Tits form of (finite and infinite) posets. We call such a form critical (with respect to positivity) if it is not positive but each of its proper subform is positive, and prove that a critical Tits form is finite (i.e. is of the Tits form of a finite poset).

The quadratic Tits form, introduced by P. Gabriel [1] for quivers, Yu.A. Drozd [2] for posets, and M.M. Kleiner, A.V. Roiter [3] and Yu.A. Drozd [4] for a wide class of classification problems, plays an important role in representation theory. In particular, there are many results on connections between representation types of various objects and properties of the Tits forms. The reader interested in this topic is refereed to the papers of [5, 6], the monographs [7, 8] and, e.g., [9–15] (with the bibliographies therein). Above all one must mention the well known result that a quiver is of finite type if and only if its Tits form is positive [1]; in the case of posets the Tits form must be weakly positive [2] (recall that representations of posets were introduced in [16]). It follows from the results of [2] that the Tits form of a poset S is weakly positive if and only if S contains no subposet isomorphic to a critical, with respect to finiteness of type, poset (critical posets are indicated in [17]; their number is 5).

Our paper is devoted to study critical, with respect to positivity, Tits forms of posets.

Formulate first the main result.

Let S be a (finite or infinite) poset and $\mathbb{Z}$ the integer numbers. Denote by $\mathbb{Z}_0^{S\cup 0}$ the subset of the cartesian product $\mathbb{Z}^{S\cup 0}$ consisting of all vectors $z = (z_i)$, $i \in S \cup 0$ with only finitely many non-zero coordinates. The quadratic Tits form of S is by definition the form $q_S : \mathbb{Z}_0^{S\cup 0} \to \mathbb{Z}$ defined by the equality

$$q_S(z) = z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j, i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i.$$

This form (as an arbitrary one) is called positive if it takes a positive value on every nonzero vector $z \in \mathbb{Z}_0^{S\cup 0}$, and non-positive if otherwise.

A quadratic form is called finite if the number of the variables is finite. In the case of the Tits form $q_S(z)$ it means that S is finite.

Denote by $\mathcal{P}$ the set of all (finite and infinite) posets and set $\mathcal{F} = \{q_S(z) \,|\, S \in \mathcal{P}\}$. A form $= q_S(z)$ from $\mathcal{F}$ is said to be critical with respect positivity (or simply critical) if f is not positive, but $f' = q_{S'}(z)$ is positive for any proper subposet S' of S.

Theorem 1. *Any critical Tits form $q_S(z)$ is finite.*

Before we prove Theorem 1, we describe infinite posets with positive Tits form.

Let S be a poset. For nonempty subsets X, Y of S, we write $X \lhd Y$ if $x < y$ for some $x \in X$, $y \in Y$, and $X \not\lhd Y$ if otherwise. We say that S is a sum of subsets A and B, and write $S = A + B$, if $A \cap B = \varnothing$ and $S = A \cup B$. If any two elements $a \in A$ and $b \in B$ are incomparable, we call this sum direct; we denote such sum by $S = A \coprod B$. The sum $S = A + B$ is called one-sided if $B \not\lhd A$ or $A \not\lhd B$ (see [18]). Finally, the sum $S = A + B$ is called minimax if $x < y$ with x and y belonging to different summands implies that x is minimal and y maximal in S (see also [18]).

For a sum S of posets A and B let $R^<(A, B)$ denotes the set of pairs $(x, y) \in A \times B$ with $x < y$. Such a pair (a, b) is called short if there is no other such a pair (a', b') satisfying $a \leqslant a'$,

$b' \leqslant b$. By $R_0^<(A,B)$ we denote the subset of all short pairs from $R^<(A,B)$. We call the order $r_0 = r_0(A,B)$ of $R_0(A,B) = R_0^<(A,B) \cup R_0^<(B,A)$ the rank of the sum S.

Note that direct sums are particular cases of sums and, in particular, one-sided sums, minimax sums, etc. (more precisely, they are sums of zero rank). We could exclude them from others, considering sums of ranks 0 and greater than 0 separately (as in [18]), but here we do not do it.

Recall that a linear ordered set is also called a chain. A poset with the only pair of incomparable elements is called an almost chain.

We have the following theorem.

Theorem 2. *Let S be an infinite poset. Then the Tits form of S is positive if and only if one of the following condition holds*:

1) *S is a direct sum of two chains;*
2) *S is a direct sum of a chain and an almost chain;*
3) *S is a one-sided minimax sum (of rank 1) of two chains.*

This theorem describes completely infinite posets with positive Tits form. It was proved by the authors in [18].

We now present the scheme of the proof of the necessity of Theorem 2, calling particular attention to critical posets.

Step 1. It is proved that the Tits form is not positive for the following posets:

$T_1 = \{1, 2, 3, 4\}$ (without comparable $i \neq j$);
$T_2 = \{1, 2, 3, 4 \,|\, 1 \prec 4, 2 \prec 4, 3 \prec 4\}$;
$T_3 = \{1, 2, 3, 4, 5, 6, 7, 8 \,|\, 2 \prec 3, 4 \prec 5 \prec 6 \prec 7 \prec 8\}$;
$T_4 = \{1, 2, 3, 4, 5, 6, 7, 8 \,|\, 1 \prec 7, 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7\}$;
$T_5 = \{1, 2, 3, 4, 5, 6, 7, 8 \,|\, 1 \prec 5 \prec 6 \prec 7 \prec 8, 2 \prec 4, 2 \prec 5, 3 \prec 4\}$;
$T_6 = \{1, 2, 3, 4 \,|\, 1 \prec 3, 1 \prec 4, 2 \prec 3, 2 \prec 4\}$;
$T_7 = \{1, 2, 3, 4, 5, 6, 7 \,|\, 1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 5\}$;
$T_8 = \{1, 2, 3, 4, 5, 6, 7 \,|\, 1 \prec 5, 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7\}$;
$T_9 = \{1, 2, 3, 4, 5, 6, 7 \,|\, 1 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 2 \prec 6\}$;
$T_{10} = \{1, 2, 3, 4, 5, 6, 7, 8 \,|\, 1 \prec 2 \prec 8, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8\}$;
$T_{11} = \{1, 2, 3, 4, 5, 6, 7, 8 \,|\, 1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 4\}$;
$T_{12} = \{1, 2, 3, 4, 5, 6, 7, 8 \,|\, 1 \prec 7, 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8\}$;
$T_{13} = \{1, 2, 3, 4, 5, 6, 7, 8 \,|\, 1 \prec 4, 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8\}$;
$T_{14} = \{1, 2, 3, 4, 5, 6, 7, 8 \,|\, 1 \prec 2 \prec 8, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 7\}$;
$T_{15} = \{1, 2, 3, 4, 5, 6, 7, 8 \,|\, 1 \prec 2 \prec 8, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 4\}$;
$T_{16} = \{1, 2, 3, 4, 5, 6, 7, 8 \,|\, 1 \prec 2 \prec 5, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 4\}$;
$T_{17} = \{1, 2, 3, 4, 5, 6, 7, 8 \,|\, 1 \prec 3 \prec 4 \prec 5 \prec 6 \prec 8, 1 \prec 2 \prec 8\}$.

It is also proved that all these posets are critical.

By $T_i^*, i = 1, \ldots, 17$, we denote the poset dual to T_i. Obviously, the posets T_i^* are also critical.

Step 2. Let S be a poset. Recall that the width of S is defined to be the maximum number $w(S)$ of pairwise incomparable elements of S. An element of S is called nodal if it is comparable to any other element. The set of all nodal elements of S is denoted by S_0.

Let S be an infinite poset and $w(S) = 2$. It is proved that, if S has no subposet isomorphic to T_i or T_i^* with $i > 5$, then the following assertions hold.

Proposition 1. *If S_0 is infinite, then S is an almost chain.*

Proposition 2. *If S_0 is empty, then S is a one-sided minimax sum of two chains.*

Proposition 3. *If S_0 is finite, but not empty, then S is a one-sided minimax sum of a chain and an almost chain.*

From these proposition it follows the necessity of Theorem 2 in the case $w(S) = 2$.

Step 3. Let S be an infinite poset and $w(S) \neq 2$. The case $w(S) = 1$ is trivial and the case $w(S) > 3$ is imposable (because the Tits form of T_1 is not positive). So it remains to consider the case $w(S) = 3$. It is proved that if S has no subposet isomorphic to T_i or T_i^* with $i > 2$, then the following assertions hold.

Proposition 4. *If R is a direct summand of S of order larger than 1, then $S \setminus R$ is an almost chain.*

Proposition 5. *S is not a one-sided minimax sum of a chain and an almost chain.*

The necessity of Theorem 2 in the case $w(S) = 3$ follows from that in the case $w(S) = 2$ and Propositions 4, 5.

It is easy to see that from the proof of the necessity of Theorem 2 it follows Theorem 1. Indeed, from that the next proposition follows.

Proposition 6. *Let S be an infinite poset with non-positive Tits form. Then S contains a subposet isomorphic to T_i or T_i^* for some $l = 1, 2, \ldots, 17$.*

And it follows from this proposition that a critical poset may not be infinite.

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