

Hidden Symmetry Exposure. The Mechanical Systems with the Hard Structure of Forces

Yuri N. BAZILEVICH

Institute of Continuing Special Education, Pridniprovsk State University of Civil Engineering, 11a Patorzhynskogo Str., 49044 Dnipropetrovsk, Ukraine

E-mail: *bazilevich@ua.fm*

The problem of the optimal decoupling of mechanical systems equations is solved. Gyroscopic and positional non-conservative forces are taken into account. Presence of the latter goes beyond the scope of compact groups theory. The way of the composition of symmetry groups corresponding to the found decoupling is considered. The method for solution of problem of hierarchic decoupling of the equations set is suggested

1 The problem of decoupling

We consider a problem of decoupling of the equations set $B_1\ddot{\mathbf{x}} + B_2\dot{\mathbf{x}} + B_3\mathbf{x} = 0$, where $\mathbf{x} \in \mathbb{C}^n$, B_i are complex (generally speaking) n by n matrices. It is necessary to find such transformation

$$\hat{B}_i = HB_iS = \text{diag}(B_{1i}, B_{2i}, \dots, B_{li}) = \begin{bmatrix} [1] & \dots & \dots & 0 \\ \dots & [2] & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & [l] \end{bmatrix},$$

that all $\{\hat{B}_i\}$ coefficient matrices will have the equal partitioned-diagonal forms, or prove that such transformation does not exist. We want to have l blocks as maximum possible number [1,2].

The second problem is similar to the first, but the matrices will have the partitioned-triangular form:

$$\hat{B}_i = \begin{bmatrix} [1] & \text{//} & \text{//} & \text{//} \\ \cdot & [2] & \text{//} & \text{//} \\ \cdot & \cdot & \cdot & \text{//} \\ 0 & \cdot & \cdot & [l] \end{bmatrix}.$$

We prove that if a B_1 matrix is nonsingular, for the solution of the problems 1 and 2 it is necessary to form auxiliary matrices $C_i = B_1^{-1}B_i$, $i = \overline{1,3}$ and for them to solve similar problems, using only similarity transformations:

$$\tilde{C}_i = R^{-1}C_iR. \tag{1}$$

2 Existing method

There exists the method of commutative matrix (see A.K. Lopatin [6], E.D. Yakubovich [9], V.V. Udilov [8] and Bazilevich [2]). Let $\Lambda(C_i)$ be a set of all matrices that are commutative with all matrices $\{C_i\}$. Let a Z matrix be a member of $\Lambda(C_i)$ and have two (or more) different eigenvalues. Vectors of its canonical basis are columns of R transformation matrix of similarity.

This method must be used at first to the parent matrices $\{C_i\}$, then to the blocks obtained consistently. We continue this process until receiving only undecoupling blocks. An application

of methods using a priori information about symmetry of the proper physical system is desirable at the first step. This carries out to get the maximal quantity of blocks. A further increase of quantity of blocks is impossible. The uniqueness theorem confirms that [2].

3 Finding a symmetry group

After implementation of decoupling it is possible to find a symmetry group of the explored physical system. Formative group members will be the matrices $T_\mu = RL_\mu R^{-1}$, where $L_\mu = \text{diag}(E_1, E_2, \dots, -E_\mu, \dots, E_l)$, E_i are identity matrices. Their orders are equal to those of corresponding matrices of the obtained subsystems.

The members of obtained set form an Abelian group as $L_j L_\mu = L_\mu L_j$ and $L_\mu^2 = E$. The group members are commutative with the $\{C_i\}$ matrices. Reducing matrices T_μ to the irreducible representations corresponds to decoupling of matrices $\{C_i\}$. In other terms this process corresponds to the similarity transformation (1).

Such group is not unique. Indeed, the given group is Abelian, while symmetry group of the physical system may be not Abelian. It is clear that the matrices of any symmetry group belong to the $\Lambda(C_i)$.

4 Hierarchic decoupling

Reducing matrices to the partitioned-triangular form corresponds to “hierarchic” (vertical) decoupling. Thus, first subsystem does not contain variables of another subsystems. Only variables of first and second subsystems are present in the following subsystem, etc. The number of such subsystems can be greater than at ordinary decoupling.

Subsystem of smaller order made from the diagonal blocks of regenerate matrices are equivalent to the initial equations set from point of dynamic stability. Indeed, a set of eigenvalues of the initial equations set is equal to a union of sets of eigenvalues of subsystems made from the diagonal blocks.

Let us prove that the number of auxiliary subsystems of only the nonsymmetrical coefficient matrices can be more than maximally possible number of independent subsystems to which the given equations set is reduced. Nonsymmetric coefficient matrices are used, for example, in the case of research of dynamic stability of the mechanical systems [5]. This is related to many phenomena: flutter, shimmy wheels of car or front undercarriage of airplane, creep caused by interaction of wheeled pair of railway vehicle with railing and so on. The asymmetry of matrix of position forces makes inapplicable many theorems of qualitative research of motion stability, and also does problems in application of methods of decoupling.

Let $\phi(C_i)$ be the algebra generated by the $\{C_i\}$ matrices. In other terms, this is a set closed with respect to addition of matrices, matrix multiplication and multiplying of matrices on numbers. The first step of the proposed method consists in construction of algebra $\phi(C_i)$. For this reason at first select linearly independent matrices among the parent matrices. We will designate them W_k ($k = \overline{1, r}$). Further we calculate all possible products $Z_{ij} = W_i W_j$. If all Z_{ij} are linear combinations of matrices $\{W_k\}$ then the latter are basic set of $\phi(C_i)$. Otherwise we add the next matrix Z_{ij} to the set $\{W_k\}$ and begin the procedure of verification of all products anew. Criterion of possibility of reducing matrices to the partitioned-triangular form (reducible algebra) is the following: algebra dimension is smaller than n^2 , where n is order of matrices.

Further we use a set of the theorems by J.H.M. Wedderburn, E. Artin, E. Noether, and others. Following those theorems a reducible algebra may be semisimple or non-semisimple [7]. In the cases in point, the given matrices are not reducible to the partitioned-diagonal form (if

only were reducible – we would already have done this by the method of commutative matrix). Therefore the algebra is not semisimple.

A non-semisimple algebra has a nontrivial radical ideal. There are computation formulas for its finding [4]. Coordinates $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_r]^T$ of any element of radical ideal in the $\{W_k\}$ basic set satisfy to the equation $D\alpha = 0$, $D = \{d_{ij}\}$, where $d_{ij} = \text{Sp}(W_i W_j)$, Sp is the trace of matrix. The intersection of kernels of all matrices of radical ideal is a nontrivial subspace. It is invariant with respect to the parent matrices. We get vectors of this base subspace and orthogonal complement to it and place these vectors as columns of S transformation matrix.

Computational algorithms and computer programs on realization of developed method were made. Computations on hierarchic decoupling of equations of motion of railway vehicles and others systems were performed [3].

5 Example

Let us consider the mechanical system consisting of two bodies (Fig. 1). Let the control device create a force $P = -x_1$, applied to the second body. There is an example of positional non-conservative force. Let $m_1 = m_2 = 1$, $k = 1$.

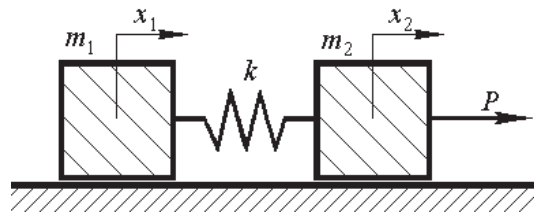


Figure 1.

Let us consider: q_1 is a moving of center of the mass; q_2 is a half of spring pressure size:

$$q_1 = (x_1 + x_2)/2, \quad q_2 = (x_1 - x_2)/2.$$

The equations of motion have such form:

$$2\ddot{q}_1 + q_1 + q_2 = 0, \quad 2\ddot{q}_2 - q_1 + 3q_2 = 0.$$

Coefficient matrices are as follows:

$$B_1 = 2E, \quad B_2 = 0, \quad B_3 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}.$$

Therefore $C_1 = E$, $C_2 = 0$, $C_3 = 0.5 \cdot \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$. We have to resolve the problem of decoupling of this system by similarity transformation.

At first, we will make sure that it is impossible to reduce the given system to the partitioned-diagonal form. For this reason we will find a Z matrix commutative with the given matrices C_i ($i = 1, 2, 3$). It has a form $Z = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha + 2\beta \end{pmatrix}$, where α, β are arbitrary parameters. So far as the Z matrix has no different eigenvalues, reducing parent matrices $\{C_i\}$ to the partitioned-triangular form is impossible [2].

In order to find if it is possible to reduce matrices $\{C_i\}$ to the partitioned-triangular form or not we will use the algorithm described above.

We consider all possible products of $C_k C_j$ matrices and will check whether the obtained matrices are the linear combination of parent matrices or not. So far as multiplication by

$C_1 = E$ does not change the matrices; it remains to consider the C_3^2 product. We calculate $C_3^2 = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$. We check whether this matrix is the linear combination of previous matrices: $C_3^2 = \alpha C_1 + \beta C_3$? This equality corresponds to the equations set:

$$\begin{aligned} 0 &= \alpha \cdot 1 + \beta \cdot 0.5, \\ 1 &= \alpha \cdot 0 + \beta \cdot 0.5, \\ -1 &= \alpha \cdot 0 - \beta \cdot 0.5, \\ 2 &= \alpha \cdot 1 + \beta \cdot 1.5. \end{aligned}$$

We obtained that $\alpha = -1$, $\beta = 2$. Consequently, matrix C_2^3 is linear combination of C_1 and C_3 matrices.

So all products belonged to the linear capsule of C_1 and C_3 matrices. Consequently, these matrices form a basic set of $\phi(C_i)$ algebra, generated by matrices $\{C_i\}$. Number r of basis elements equal to 2, that is $r < n^2 \equiv 2^2$. This means that reducing to the triangular form is possible.

We make a $D = \{\text{Sp}(C_j C_k)\}$ matrix. All products $C_j C_k$ are already calculated. We get

$$D = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

We make the system of $Dy = 0$ equations:

$$2y_1 + 2y_2 = 0, \quad 2y_1 + 2y_2 = 0.$$

As a result we get: $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. We calculate a G matrix:

$$G = 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 1 \cdot 1.5 \cdot \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = 0.5 \cdot \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

$G\xi = 0$ equations have such kind:

$$\xi_1 - \xi_2 = 0, \quad \xi_1 - \xi_2 = 0.$$

We obtained that the basis in the set of solutions of this system consists of $\mathbf{s}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ vector. This vector and $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ vector are linearly independent. Therefore $R = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Further

$$\begin{aligned} R^{-1} &= \frac{1}{2} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \tilde{C}_1 = R^{-1}ER = E \\ \tilde{C}_3 &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot 0.5 \cdot \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \cdot 0.5 \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

After transformations the initial system is led to the subsystems:

$$1) \quad \ddot{y}_2 + y_2 = 0, \quad 2) \quad \ddot{y}_1 + y_1 + y_2 = 0.$$

Note that G_1 group of matrices $g_k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, $k \in \mathbb{N}$ is symmetry group of matrices $\{\tilde{C}_i\}$.

This group is not compact. Indeed, the subspace $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ (where α is any number) is an invariant subspace in relation to G_1 group and has no direct object invariant with respect to G_1 group.

6 Extensions

A developed method may be applied to equations of evolution of the automatic control systems and to the models of macroeconomics.

- [1] Bazilevich Y.N., Reduction of system of linear differential equations to the maximal number of independent subsystems, *Diff. Uravn.*, 1980, V.16, N 2, 360–361 (in Russian).
- [2] Bazilevich Y.N., Numerical decoupling methods in the linear problems of mechanics, Kyiv, Naukova Dumka, 1987 (in Russian).
- [3] Bazilevich Y.N., Korotenko M.L. and Shvets I.V., Solving the problem on hierarchical decoupling the linear mathematical models of mechanical systems, *Tekhnicheskaya Mekhanika*, 2003, N 1, 135–141 (in Russian).
- [4] Chebotariov N.G., Introduction to theory of algebras, Moscow – Leningrad, Gostekhteorizdat, 1949 (in Russian).
- [5] Lazarjan V.A., Dlugach L.A. and Korotenko M.L., Dynamic stability of railway vehicle, Kyiv, Naukova Dumka, 1972 (in Russian).
- [6] Lopatin A.K., The algebraic reducibility of systems of linear differential equations. I, *Diff. Uravn.*, 1968, V.4, 439–445; The algebraic reducibility of systems of linear differential equations on a finite interval of definition of the independent variable. II, *Diff. Uravn.*, 1968, V.4, 839–844 (in Russian).
- [7] Pierce R.S., Associative algebras, New York, Springer-Verlag, 1982.
- [8] Udilov V.V., Application of methods of abstract algebra to investigation of multidimensional automatic control systems, *Kibernetika i Vichisl. Tekhnika*, 1974, V.23, 20–27 (in Russian).
- [9] Yakubovich E.D., Construction of replacement systems for a class of multidimensional linear automatic control systems, *Izv. Vuzov, Radiofizika*, 1969, V.12, N 3, 362–377 (in Russian).