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Enhanced Binding in a Model of an Abstract Quantum System Coupled to a Multi-Component Bose Field

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A model of an abstract quantum system coupled to a multi-component Bose field is considered. Under suitable hypotheses, the model has a ground state even if the uncoupled (zero-coupling) system has no ground state.

1 Introduction

A quantum system which has no ground states may be changed, through coupling to a quantum field, to have a ground state. If such a change occurs, then we say that the coupled system has *enhanced binding* with respect to ground state. The phenomenon of enhanced binding, if it occurs, may be regarded as one of the evidences supporting the view point that quantum fields are more fundamental objects underlying the material world.

From this point of view, as well as from a purely mathematical one, it is interesting to clarify whether or not enhanced binding *indeed* occurs in models of a quantum system – typically a system of nonrelativistic quantum particles – coupled to a quantum field.

The problem of enhanced binding was first discussed by Hiroshima and Spohn [11]. They discussed the Pauli–Fierz model in nonrelativistic quantum electrodynamics in the dipole approximation and proved that, under suitable hypotheses, enhanced binding occurs for large coupling constants. Hainzl, Vougalter and Vugalter [10] considered the Pauli–Fierz model without the dipole approximation showing that it has enhanced binding for small coupling constants. The results and the methods in [10] have been extended to the Pauli–Fierz model with spin [7,8] (cf. also [9]).

In a previous paper [6] the enhanced binding problem was considered for a general class of quantum field models, called the generalized spin-boson (GSB) model which describes an abstract quantum system coupled linearly to a Bose field [3–5], and proved, under suitable hypotheses, the existence of enhanced binding for a region of coupling constants. The GSB model was extended to a more general one in [2], whose Hamiltonian is obtained by adding quadratic self-interaction terms of the Bose field to the Hamiltonian of the GSB model, and it was shown that results similar to those in [2] hold also in the extended GSB model.

In this paper we consider a slightly more general model than the extended GSB model in [2] and show that, under suitable hypotheses, enhanced binding occurs in this model too.

The present paper is organized as follows. Section 2 is a preliminary section which recalls basic objects and elementary facts in the theory of the abstract boson Fock space. In Section 3 we define quadratic operators acting in the abstract boson Fock space. In Section 4 we describe the model considered in the present paper. The main theorems are stated in Section 5. The last section is devoted to sketches of proofs of them.

2 Bose fields

We denote the inner product and the norm of a Hilbert space $\mathcal{X}$ by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\| \cdot \|_{\mathcal{X}}$ respectively, where we use the convention that the inner product is antilinear (resp. linear) in the first (resp.

second) variable. We sometimes omit the subscript $\mathcal{X}$ in $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{X}}$, if there is no danger of confusion.

For a linear operator T on a Hilbert space, we denote its domain by $D(T)$. For a subspace $D \subset D(T)$, $T|D$ denotes the restriction of T to D. If T is densely defined, then the adjoint of T is denoted T^*. For linear operators S and T on a Hilbert space, $D(S+T) := D(S) \cap D(T)$ unless otherwise stated.

For each complex Hilbert space $\mathcal{X}$, the boson Fock space $\mathcal{F}_{\rm b}(\mathcal{X})$ over $\mathcal{X}$ is defined by

$$\mathcal{F}_{\rm b}(\mathcal{X}) := \oplus_{n=0}^{\infty} \otimes_{\rm s}^{n} \mathcal{X},$$

where $\otimes_{\rm s}^{n}\mathcal{X}$ denotes the n-fold symmetric tensor product of $\mathcal{X}$ with $\otimes_{\rm s}^{0}\mathcal{X} := \mathbb{C}$ (the set of complex numbers).

The annihilation operator $a(f)$ ($f \in \mathcal{X}$) on $\mathcal{F}_{\rm b}(\mathcal{X})$ is defined to be a densely defined closed linear operator such that, for all $\psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in D(a(f)^*)$, $(a(f)^*\psi)^{(0)} = 0$ and

$$(a(f)^*\psi)^{(n)} = \sqrt{n} S_n \left(f \otimes \psi^{(n-1)} \right), \qquad n \geq 1,$$

where S_n is the symmetrization operator on $\otimes^n \mathcal{X}$. The adjoint $a(f)^*$, called the creation operator, and the annihilation operator $a(g)$ ($g \in \mathcal{X}$) obey the canonical commutation relations

$$[a(f), a(g)^*] = \langle f, g\rangle_{\mathcal{X}}, \qquad [a(f), a(g)] = 0, \qquad [a(f)^*, a(g)^*] = 0$$

for all $f, g \in \mathcal{X}$ on the dense subspace

$$\mathcal{F}_0(\mathcal{X}) := \{\psi \in \mathcal{F}_{\rm b}(\mathcal{X}) \,|\, \text{there exists a number } n_0 \text{ such that } \psi^{(n)} = 0 \text{ for all } n \geq n_0\},$$

where $[X, Y] := XY - YX$.

We denote by $\Omega := \{1, 0, 0, \ldots\}$ the Fock vacuum in $\mathcal{F}_{\rm b}(\mathcal{X})$. For a subspace $\mathcal{D} \subset \mathcal{X}$ we define a subspace

$$\mathcal{F}_{\rm fin}(\mathcal{D}) := \mathcal{L}\left(\{\Omega,\, a(f_1)^* \cdots a(f_n)^*\Omega \,|\, n \in \mathbb{N}, f_j \in \mathcal{D}, j = 1, \ldots, n\}\right),$$

where $\mathcal{L}(\{\cdot\})$ means the subspace algebraically spanned by the set $\{\cdot\}$. If $\mathcal{D}$ is dense in $\mathcal{X}$, then the subspace $\mathcal{F}_{\rm fin}(\mathcal{D})$ is dense in $\mathcal{F}_{\rm b}(\mathcal{X})$.

Let

$$\phi(f) := \frac{a(f) + a(f)^*}{\sqrt{2}}, \qquad f \in \mathcal{X},$$

which is called the Segal field operator. It is shown that $\phi(f)$ is essentially self-adjoint on $\mathcal{F}_0(\mathcal{X})$ [13, § X.7]. *We denote its closure by the same symbol* $\phi(f)$. The "conjugate momentum" of $\phi(f)$ is defined by

$$\pi(f) := \phi(if), \qquad f \in \mathcal{X}.$$

We have

$$[\phi(f), \pi(g)] = i\mathrm{Im}\left(i\langle f, g\rangle_{\mathcal{X}}\right).$$

For every symmetric operator S on $\mathcal{X}$, one can define a closed symmetric operator $d\Gamma(S)$, called the second quantization of S, by

$$d\Gamma(S) := \oplus_{n=0}^{\infty} S^{(n)},$$

with $S^{(0)} = 0$ and $S^{(n)}$ is the closure of

$$\left(\sum_{j=1}^{n} I \otimes \cdots \otimes \overset{j\text{th}}{\breve{S}} \otimes \cdots \otimes I \right) \Bigg| \otimes_{\rm alg}^{n} D(S),$$

where I denotes identity and $\otimes_{\rm alg}^{n}$ algebraic tensor product. If S is self-adjoint, then so is $d\Gamma(S)$.

3 Quadratic operators

In the boson Fock space $\mathcal{F}_{\mathrm{b}}(\mathcal{X})$, one can define, in a general manner, linear operators quadratic in the annihilation and the creation operators. Let C be a conjugation on $\mathcal{X}$, i.e., C is an anti-linear, isometric mapping on $\mathcal{X}$ such that $C^2 = I$. Let K be a Hilbert–Schmidt operator on $\mathcal{X}$. Then there exist orthonormal sets $\{f_n\}_{n=1}^M$, $\{g_n\}_{n=1}^M$ in $\mathcal{X}$ and a sequence $\{\lambda_n\}_{n=1}^M$ of positive numbers ($M < \infty$ or M is countably infinite) such that $\sum_{n=1}^M \lambda_n^2 < \infty$ and

$$K = \sum_{n=1}^M \lambda_n \langle f_n, \cdot \rangle_{\mathcal{X}} g_n,$$

where, in the case $M = \infty$, the sum on the right hand side converges in operator norm [12, Theorems VI.17 and VI.22]. Using this representation of K, we define linear operators $(a^*|K|a^*)_N$ and $(a|K|a)_N$ ($N \in \mathbb{N}$) acting in $\mathcal{F}_{\mathrm{b}}(\mathcal{X})$, with domain $\mathcal{F}_{\mathrm{fin}}(\mathcal{X})$, by

$$(a^*|K|a^*)_N := \sum_{n=1}^{N\wedge M} \lambda_n a(Cf_n)^* a(g_n)^*, \qquad (a|K|a)_N := \sum_{n=1}^{N\wedge M} \lambda_n a(f_n) a(Cg_n).$$

where $N \wedge M := \min\{N, M\}$ if $M < \infty$ and $N \wedge M := N$ if $M = \infty$. It is easy to see that, for all $\Psi \in \mathcal{F}_{\mathrm{fin}}(\mathcal{X})$, the strong limit

$$(a^\#|K|a^\#)\Psi := \text{s-}\lim_{N\to\infty} (a^\#|K|a^\#)_N \Psi$$

exists, where $a^\#$ denotes either a^* or a. Moreover, the operator $(a^\#|K|a^\#)$ with domain $\mathcal{F}_{\mathrm{fin}}(\mathcal{X})$ is closable and

$$(a^*|K|a^*)^* \supset (a|K^*|a), \qquad (a|K|a)^* \supset (a^*|K^*|a^*).$$

4 Definition of the model

We consider a model of an abstract quantum system S coupled to an N-component Bose field over $\mathbb{R}^d$ ($d, N \in \mathbb{N}$). We denote the Hilbert space of the system S by $\mathcal{H}$, which is taken to be an arbitrary separable complex Hilbert space. In concrete realizations, S may be a system of nonrelativistic quantum particles or a quantum field system.

The one-particle Hilbert space of the Bose field is taken to be

$$\mathcal{M} := \oplus^N L^2(\mathbb{R}^d),$$

the N direct sum of $L^2(\mathbb{R}^d)$. Then the Hilbert space for the Bose field is given by the Fock space $\mathcal{F}_{\mathrm{b}}(\mathcal{M})$ over $\mathcal{M}$.

Let μ be a Borel measurable function on $\mathbb{R}^d$ such that $0 < \mu(k) < \infty$ for almost everywhere (a.e.) $k \in \mathbb{R}^d$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Then μ defines a multiplication operator on $L^2(\mathbb{R}^d)$, which is nonnegative, injective and self-adjoint. We denote it by the same symbol. We define an operator

$$\widehat{\mu} := \oplus^N \mu$$

acting in $\mathcal{M}$.

The Hilbert space of the coupled system of S and the Bose field is given by the tensor product

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{M}).$$

Let A be a self-adjoint operator on $\mathcal{H}$, which denotes physically the Hamiltonian of the quantum system S.

We say that a densely defined linear operator on a Hilbert space is Hilbert–Schmidt if it is bounded and its closure is Hilbert–Schmidt. We denote the closure by the same symbol.

In what follows, we denote by C the complex conjugation on $\mathcal{M}$. For a bounded linear operator T on $\mathcal{M}$, we define T_C by

$$T_C := CTC.$$

Let S and T be bounded linear operators on $\mathcal{M}$. We assume the following.

Hypothesis (I).

(I.1) The operators $T\widehat{\mu}S^*$ and $T\widehat{\mu}^{1/2}$ are densely defined and Hilbert–Schmidt.

(I.2) The operator $S\widehat{\mu}S^* + T_C\widehat{\mu}T_C^*$ is densely defined.

Then we can define an operator $H_{\mathrm{b}}(\mu, S, T)$ by

$$H_{\mathrm{b}}(\mu, S, T) := d\Gamma(S\widehat{\mu}S^* + T_C\widehat{\mu}T_C^*) + (a|T\widehat{\mu}S^*|a) + (a|T\widehat{\mu}S^*|a)^*.$$

It is easy to see that $H_{\mathrm{b}}(\mu, S, T)$ is a symmetric operator with

$$D(H_{\mathrm{b}}(\mu, S, T)) \supset \mathcal{F}_{\mathrm{fin}}\left(D(S\widehat{\mu}S^*) \cap D(T_C\widehat{\mu}T_C^*)\right).$$

We denote the closure of $H_{\mathrm{b}}(\mu, S, T)$ by $\bar{H}_{\mathrm{b}}(\mu, S, T)$.

The Hamiltonian of the model we consider in the present paper is defined by

$$H := A \otimes I + I \otimes \bar{H}_{\mathrm{b}}(\mu, S, T) + \sum_{j=1}^{J} B_j \otimes \phi(g_j) + \sum_{j=1}^{J} K_j \otimes \pi(h_j),$$

where B_j ($j = 1, \ldots, J$; $J \in \mathbb{N}$) is a symmetric operator on $\mathcal{H}$ such that $\cap_{j=1}^{J} D(B_j)$ is dense in $\mathcal{H}$, K_j ($j = 1, \ldots, J$) is a bounded self-adjoint operator on $\mathcal{H}$ and $g_j, h_j \in \mathcal{M}$, $j = 1, \ldots, J$.

Remark 1. (i) The operator H with $T = 0$, $h_j = 0$ (or $K_j = 0$) and $S\widehat{\mu}S^* = \widehat{\omega} := \oplus^N \omega$ (ω is a nonnegative Borel measurable function on $\mathbb{R}^d$) yields

$$H_{\mathrm{AH}} := A \otimes I + I \otimes d\Gamma(\widehat{\omega}) + \sum_{j=1}^{J} B_j \otimes \phi(g_j).$$

This is the Hamiltonian of the GSB model [3]. The existence of ground states of H_{AH} with $N = 1$ was discussed in [3] under the assumption that A has a ground state (cf. also [4] for further extensions). The problem of enhanced binding in the GSB model was considered in [6]. For the absence of ground states of H_{AH}, see [5].

(ii) The problem of enhanced binding in the model H with $h_j = 0$ ($j = 1, \ldots, J$) was discussed in [2]. The results presented below are extensions of those in [2] to the case where $h_j \neq 0$.

Example 1. The Pauli–Fierz Hamiltonian in the dipole approximation [11] is a special case of H with $d = 3$, $N = 2$, $J = 3$. In this sense our model is an abstract generalization of the Pauli–Fierz model in the dipole approximation.

5 Main results

For a self-adjoint operator L on a Hilbert space, we denote its spectrum (resp. essential spectrum) by $\sigma(L)$ (resp. $\sigma_{\mathrm{ess}}(L)$).

Definition 1. Let L be a self-adjoint operator on a Hilbert space bounded from below and set

$$E_0(L) := \inf \sigma(L),$$

which is called the ground state energy of L. We say that L has a ground state if $E_0(L)$ is an eigenvalue of L. In that case, each non-zero vector in $\ker(L - E_0(L))$ is called a *ground state* of L.

To state the main results of this paper, we formulate additional hypotheses. For this purpose, we first recall an important notion on commutativity of self-adjoint operators:

Definition 2. We say that two self-adjoint operators S_1 and S_2 on a Hilbert space strongly commute (or S_1 strongly commutes with S_2) if their spectral measures commute.

A family of self-adjoint operators $\{S_j\}_{j=1}^n$ on a Hilbert space is said to be strongly commuting if S_j strongly commutes with S_l for all $j, l = 1, \ldots, n$ with $j \neq l$.

In what follows, we assume that A is of the form

$$A = A_0 + A_1$$

with A_0 a nonnegative self-adjoint operator and A_1 a symmetric operator on $\mathcal{H}$.

We introduce $\tilde{g}_j, \tilde{h}_j \in \mathcal{M}$ $(j = 1, \ldots, J)$ by

$$\tilde{g}_j := S^* g_j - T^* C g_j, \qquad \tilde{h}_j := S^* h_j - T^* C h_j,$$

where S, T and C are operators introduced in the preceding section.

Hypothesis (II). $\tilde{g}_j$, $\tilde{g}_j/\mu^{3/2}, \tilde{h}_j, \tilde{h}_j/\mu \in \mathcal{M}$ $(j = 1, \ldots, J)$ and

$$\langle \tilde{g}_j(k), \tilde{g}_l(k)\rangle_{\mathbb{C}^N}, \langle \tilde{g}_j(k), \tilde{h}_l(k)\rangle_{\mathbb{C}^N} \in \mathbb{R}, \text{ a.e. } k \in \mathbb{R}^d \ (j, l = 1, \ldots, J).$$

Remark 2. Hypothesis (II) implies that (i) $\{\phi(i\widetilde{g}_j/\mu)\}_{j=1}^J$ is a family of strongly commuting self-adjoint operators and each $\phi(i\widetilde{g}_j/\mu)$ strongly commutes with each $\pi(\widetilde{h}_l)$ $(j, l = 1, \ldots, J)$; (ii) $[\phi(\tilde{g}_j), \pi(\tilde{h}_l)] = i\langle \tilde{g}_j, \tilde{h}_l\rangle_{\mathcal{M}}$ on $\mathcal{F}_0(\mathcal{M})$.

Hypothesis (III). The operator A_1 is A_0-bounded, i.e., $D(A_0) \subset D(A_1)$ and there exist constants $a, b \geq 0$ such that, for all $u \in D(A_0)$,

$$\|A_1 u\|_{\mathcal{H}} \leq a\|A_0 u\|_{\mathcal{H}} + b\|u\|_{\mathcal{H}}.$$

Hypothesis (IV). The operator A_0 strongly commutes with each B_j $(j = 1, \ldots, J)$ and

$$D(A_0) \subset \cap_{j,l=1}^J D(B_j B_l).$$

Moreover, there exist constants $c_j, d_j \geq 0$ such that, for all $u \in D(A_0^{1/2})$,

$$\|B_j u\|_{\mathcal{H}} \leq c_j \|A_0^{1/2} u\|_{\mathcal{H}} + d_j \|u\|_{\mathcal{H}} \qquad (j = 1, \ldots, J).$$

Hypothesis (V). The set $\{B_j\}_{j=1}^J$ is a family of strongly commuting self-adjoint operators.

Hypothesis (VI). $D(A_0) \subset \cap_{j=1}^J D(B_j A_1) \cap D(A_1 B_j)$ and $[B_j, A_1]|D(A_0)$ is bounded ($j = 1, \dots, J$). We denote the operator norm of $[B_j, A_1]$ by $\|[B_j, A_1]\|$.

We introduce an operator

$$R_B := \frac{1}{2} \sum_{j,l=1}^{J} \left\langle \frac{\tilde{g}_j}{\sqrt{\mu}}, \frac{\tilde{g}_l}{\sqrt{\mu}} \right\rangle_{\mathcal{M}} B_j B_l.$$

and define

$$A_{\mathrm{ren}} := A - R_B.$$

Under Hypotheses (II)–(IV), we have $D(A_{\mathrm{ren}}) = D(A_0)$.

Hypothesis (VII). The operator A_{ren} is self-adjoint and bounded from below.

Hypothesis (VIII).
(VIII.1) The operator T is Hilbert–Schmidt.
(VIII.2) The operators S and T satisfy the following relations:

$$S^* S - T^* T = I, \qquad SS^* - T_C T_C^* = I, \qquad ST^* = T_C S_C^*, \qquad S^* T_C = T^* S_C.$$

One can prove the following fact:

Theorem 1. *Assume Hypotheses* (I)–(VIII)*. Then H is self-adjoint and bounded from below.*

We set

$$\mu_0 := \mathrm{ess.} \inf_{k \in \mathbb{R}^d} \mu(k),$$

where ess. inf means essential infimum.

Theorem 2. *Assume Hypotheses* (I)–(VIII)*. Suppose that*

$$\{\mu(k) | k \in \mathbb{R}^d\} = [\mu_0, \infty) \tag{1}$$

Then the following (i) *and* (ii) *hold.*
(i) *If $\mu_0 > 0$, then $[E_0(H) + \mu_0, \infty) \subset \sigma_{\mathrm{ess}}(H)$.*
(ii) *If $\mu_0 = 0$, then $\sigma(H) = [E_0(H), \infty)$.*

To establish an existence theorem of a ground state of H *without* the assumption that A has a ground state, we need additional conditions.

Hypothesis (IX). The function μ is continuous on $\mathbb{R}^d$ with

$$\lim_{|k| \to \infty} \mu(k) = \infty$$

and there exist constants $\gamma > 0$ and $c_0 > 0$ such that

$$|\mu(k) - \mu(k')| \le c_0 |k - k'|^\gamma (1 + \mu(k) + \mu(k')), \qquad k, k' \in \mathbb{R}^d.$$

For $s \ge 0$, we introduce constants $C_s(\tilde{g}), D_s(\tilde{h})$ ($\tilde{g} := (\tilde{g}_1, \dots, \tilde{g}_J), \tilde{h} := (\tilde{h}_1, \dots, \tilde{h}_J)$) by

$$C_s(\tilde{g}) := \sqrt{2} \sum_{j=1}^{J} \|[B_j, A_1]\| \left\| \frac{\tilde{g}_j}{\mu^s} \right\|_{\mathcal{M}}, \qquad D_s(\tilde{h}) := \sqrt{2} \sum_{j=1}^{J} \|K_j\| \left\| \frac{\tilde{h}_j}{\mu^s} \right\|_{\mathcal{M}}.$$

Provided that $\tilde{g}_j/\mu^s \in \mathcal{M}$ and $\tilde{h}_j/\mu^s \in \mathcal{M}$ $(j = 1, \ldots, J)$ respectively. We define constants F_α $(\alpha = 1, 2, 3)$ by

$$F_1 := C_1(\tilde{g}) + D_0(\tilde{h}), \qquad F_2 := C_2(\tilde{g}) + \frac{1}{2}D_1(\tilde{h}), \qquad F_3 := C_{3/2}(\tilde{g}) + D_{1/2}(\tilde{h}).$$

We set

$$\Sigma(A_{\mathrm{ren}}) := \inf \sigma_{\mathrm{ess}}(A_{\mathrm{ren}}).$$

Generally speaking, the existence of a ground state of H may depend on whether μ_0 is positive or zero [5]. We first state a result on the existence of enhanced binding in the case $\mu_0 > 0$.

Theorem 3. (Enhanced binding in the case $\mu_0 > 0$). *Consider the case $\mu_0 > 0$. Assume Hypotheses* (I)–(IX) *and that*

$$\Sigma(A_{\mathrm{ren}}) - E_0(A_{\mathrm{ren}}) > \mu_0 + \frac{1}{2}F_3^2 + F_1. \tag{2}$$

Then H has purely discrete spectrum in the interval $[E_0(H), E_0(H) + \mu_0)$. In particular, H has has a ground state.

Remark 3. Condition (2) implies that $E_0(A_{\mathrm{ren}})$ is a discrete eigenvalue of A_{ren} and hence A_{ren} has a finite number of ground states. But A *does not necessarily have a ground state.*

Corollary 1. *Under the assumption of Theorem* 3 *and condition* (1),

$$\sigma_{\mathrm{ess}}(H) = [E_0(H) + \mu_0, \infty).$$

Theorem 4. (Enhanced binding in the case $\mu_0 = 0$). *Consider the case $\mu_0 = 0$. Assume Hypotheses* (I)–(IX) *with $\tilde{g}_j/\mu^2 \in \mathcal{M}$ $(j = 1, \ldots, J)$ in addition. Suppose that*

$$\Sigma(A_{\mathrm{ren}}) - E_0(A_{\mathrm{ren}}) > \frac{1}{2}F_3^2 + F_1. \tag{3}$$

and

$$\frac{F_1^2}{[\Sigma(A_{\mathrm{ren}}) - E_0(H)]^2} + \left\{ \frac{2F_1^2}{[\Sigma(A_{\mathrm{ren}}) - E_0(H)]^2} + 1 \right\} \frac{1}{2}F_2^2 < 1. \tag{4}$$

Then H has a ground state.

Remark 4. In Theorems 3 and 4, *the existence of a ground state of A is not assumed.*

6 Proofs of the main theorems

We give only sketches of proofs of the main theorems stated in the preceding section.

6.1 Proof of Theorem 1

Lemma 1. *There exists a unitary operator W on $\mathcal{F}_{\mathrm{b}}(\mathcal{M})$ such that*

$$(I \otimes W)H(I \otimes W)^{-1} = H_1 - \|T\widehat{\mu}^{1/2}\|_{\mathrm{HS}}^2,$$

where

$$H_1 := A \otimes I + I \otimes d\Gamma(\widehat{\mu}) + \sum_{j=1}^{J} B_j \otimes \phi(\tilde{g}_j) + \sum_{j=1}^{J} K_j \otimes \pi(\tilde{h}_j)$$

and $\|\cdot\|_{\mathrm{HS}}$ denotes Hilbert–Schmidt norm.

Proof. Similar to the proof of [2, Lemma 12]. ■

We introduce a unitary operator

$$U := \prod_{j=1}^{J} e^{-iB_j \otimes \phi(i\tilde{g}_j/\mu)}.$$

Let

$$H_0 := A_{\mathrm{ren}} \otimes I + I \otimes d\Gamma(\widehat{\mu}),$$

$$V_1 := U(A_1 \otimes I)U^{-1} - A_1 \otimes I, \qquad V_2 := \sum_{j=1}^{J} \left(U(K_j \otimes I)U^{-1}\right) I \otimes \pi(\tilde{h}_j).$$

and

$$\widetilde{H}_1 := H_0 + V_1 + V_2.$$

Lemma 2. *Assume Hypotheses* (I)–(VIII). *Then* $UD(H_0) = D(H_0)$ *and, for all* $\Psi \in D(H_0)$,

$$UH_1U^{-1}\Psi = \widetilde{H}_1\Psi.$$

Proof. Similar to the proof of [6, Lemma 3.7]. ■

Using [6, Lemma 3.10] and the well-known estimates

$$\|a(f)\Psi\| \le \left\|\frac{f}{\sqrt{\mu}}\right\|_{\mathcal{M}} \|d\Gamma(\widehat{\mu})^{1/2}\Psi\|, \qquad \|a(f)^*\Psi\| \le \left\|\frac{f}{\sqrt{\mu}}\right\|_{\mathcal{M}} \|d\Gamma(\widehat{\mu})^{1/2}\Psi\| + \|f\|_{\mathcal{M}}\|\Psi\|$$

holding for all $\Psi \in D\left(d\Gamma(\widehat{\mu})^{1/2}\right)$ and $f, f/\sqrt{\mu} \in \mathcal{M}$, one can easily see that V_1 and V_2 are infinitesimally small with respect to H_0. Hence, by the Kato–Rellich theorem, $\widetilde{H}_1$ is self-adjoint with $D(\widetilde{H}_1) = D(H_0)$ and bounded from below. By this fact and Lemma 2, H_1 is self-adjoint with $D(H_1) = D(H_0)$ and bounded from below. Theorem 1 now follows from this fact and Lemma 1.

6.2 Proof of Theorem 2

By the self-adjointness of H (Theorem 1), H_1 and Lemma 1, we have

$$\sigma(H) = \sigma(H_1) - \|T\widehat{\mu}^{1/2}\|_{\mathrm{HS}}^2, \qquad \sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_1) - \|T\widehat{\mu}^{1/2}\|_{\mathrm{HS}}^2.$$

In particular

$$E_0(H) = E_0(H_1) - \|T\widehat{\mu}^{1/2}\|_{\mathrm{HS}}^2.$$

On the other hand, one can show in the same way as in [1, Theorem 3.3],

$$[E_0(H_1) + \mu_0, \infty) \subset \sigma_{\mathrm{ess}}(H_1) \qquad (\mu_0 > 0)$$

and

$$\sigma(H_1) = [E_0(H_1), \infty) \qquad (\mu_0 = 0).$$

These facts imply Theorem 2.

6.3 Proofs of Theorems 3 and 4

By Lemma 1 and Theorem 1, it is sufficient to prove that H_1 or $\widetilde{H}_1$ has a ground state. This is done by using Lemma 2. Indeed, one sees that the methods developed in [6] work in the present case too (in [6], $\widetilde{H}_1$ with $V_2 = 0$ is considered). This is due to the fact that the new perturbation term V_2 has properties similar to those of V_1, e.g.,

$$\|V_2\Psi\| \leq D_{1/2}\|I \otimes d\Gamma(\widehat{\mu})^{1/2}\Psi\| + \frac{1}{2}D_0\|\Psi\|, \qquad \Psi \in D\left(I \otimes d\Gamma\left(\widehat{\mu}\right)^{1/2}\right),$$

$$[V_2, I \otimes a(f)]\Phi = -\frac{i}{\sqrt{2}}\sum_{j=1}^{J} U(K_j \otimes I)U^{-1}\langle f, \tilde{h}_j\rangle_{\mathcal{M}}\Phi, \qquad \Phi \in D(I \otimes N_{\rm b}),$$

where $N_{\rm b} := d\Gamma(I)$ is the number operator on $\mathcal{F}_{\rm b}(\mathcal{M})$. It turns out that we need only to shift the constants $c_s(g)$ $(s = 1, 3/2, 2)$ used in Theorems 2.2 and 2.3 in [6], which yields conditions (2)–(4) in the present context.

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