Nonlinear Symmetries of Variational Calculus and Regularity Properties of Differential Flows on Non-Compact Manifolds

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This talk is devoted to the discussion of symmetries that arise in the high-order variational equations. Such symmetries lead to a new class of nonlinear estimates on regularity. They permit the work in the case of essentially non-Lipschitz nonlinear differential equations, i.e. when the classical Cauchy–Liouville–Picard regularity scheme fails to work. These estimates are applied to the problem of C^∞ regularity of nonlinear differential flows on manifolds, that could also contain random terms. In particular, we demonstrate that the geometrically correct study of regularity problems for nonlinear flows on manifolds requires introduction of a new type variations with respect to the initial data. These variations are defined via a natural generalization of covariant Riemannian derivatives. We also find how the curvature manifests in the structure of the high-order variational equations.

1 Nonlinear differential equation on manifold with random terms

Origination of different type nonlinearities in differential equations often complicates the study of regular dependence of solutions on the initial data and parameters. The well-known classical regularity scheme, usually attributed to Cauchy, Liouville and Picard, permits to obtain a complete picture of regularity properties for quasi-linear equations (with globally Lipshitz coefficients with all bounded derivatives). For a simple one-dimensional equation

$$y_t(x) = x + \int_0^t F(y_s(x))ds \tag{1}$$

it applies the fixed point techniques for contractive for small t mapping $x \to y_t^x$ to prove the existence and uniqueness of solutions. In a similar way, the application of results on differentiability of implicit function leads to the C^∞ dependence of solution y_t^x on initial data x.

In the case of essentially nonlinear equations with coefficients with unbounded derivatives this scheme does not work. Moreover, the traditional attempt to find good Lipschitz approximations may fail due to the influence of geometry or infinite dimensions. In this talk we demonstrate that the knowledge of *symmetries of high-order differential calculus* permits to work with regularity problems in the essentially nonlinear case. Main attention is devoted to the influence of geometry on regularity properties.

We consider stochastic generalization of equation (1)

$$y_t^x = x + \int_0^t A_0(y_s^x)ds + \sum_\alpha \int_0^t A_\alpha(y_t^x)\delta W_t^\alpha \tag{2}$$

on non-compact smooth oriented connected Riemannian manifold M without boundary. Here coefficients A_0, A_α represent smooth globally defined vector fields, W_t^α denotes a family of one dimensional independent Wiener processes, δW^α means Stratonovich differential, range of index α corresponds to the dimension of manifold.

We intentionally use uncommon for stochastics notation y_t^x for the process, because the results are still valid for the ordinary differential equations ($A_\alpha = 0$). The new geometric effects also arise for this case, i.e. do not follow from the traditional arguments, previously developed in attempts to construct the stochastic differential geometry, e.g. [4–10].

Equation (2) is understood in a sense, that for any C^3 function on manifold equation

$$f(y_t^x) = f(x) + \int_0^t (A_0 f)(y_s^x)ds + \sum_\alpha \int_0^t (A_\alpha f)(y_t^x)\delta W_t^\alpha$$

holds as stochastic equation in $\mathbb{R}^1$. In particular, one can take $f(x) = x^k$ to find its local coordinates representations.

The corresponding semigroup

$$(e^{-tH}f)(x) = \mathbf{E}\, f(y_t^x) \tag{3}$$

provides the solutions to the parabolic Cauchy problem

$$\frac{\partial}{\partial t}u(t,x) + Hu(t,x) = 0, \qquad H = -\frac{1}{2}\sum_\alpha A_\alpha A_\alpha + A_0 \tag{4}$$

and gives therefore a set of actual applications to the problems of infinite dimensional functional and nonlinear analysis, stochastics and mathematical physics, differential geometry and operator theory. We remark that paper can be read in a simpler nonstochastic case for $A_\alpha = 0$.

2 Nonlinear symmetries of differential calculus

The main idea of work in the essentially nonlinear case is to find a set of *a priori estimates on regularity*. They are related with symmetries of high-order derivatives [1].

Let us consider some first order differentiation d on the algebra of states y. Then the high-order differential of nonlinear functional is given by

$$d^n f(y) = f'(y)d^n y + \sum_{j_1 + \cdots + j_s = n,\ s=2,n-1} f^{(s)}(y)d^{j_1}y\cdots d^{j_s}y + f^{(n)}(y)[dy]^n.$$

The main observation is that above terms $d^n y \sim [dy]^n$ arise simultaneously. This symmetry also manifests in the intermediate terms $d^{j_1}y\cdots d^{j_s}y \sim [dy]^{j_1+\cdots+j_s} \sim [dy]^n$ and is present in *all* differentials $d^m f$, $m \in \mathbb{N}$.

Returning to equation (1) and writing equations on high-order variations one gets

$$y_t^{(n)} = \partial_x^{(n)}y_t^x = \partial_x^{(n)}x + \int_0^t \partial_x^{(n)}F(y_s^x)ds = \partial_x^{(n)}x + \int_0^t \sum_{\substack{j_1 + \cdots + j_s = n \\ s = 1, \ldots, n}} F^{(s)}(y_s^x)y_t^{(j_1)}\cdots y_t^{(j_s)}ds.$$

So the high-order variation $y_t^{(n)}$ in the l.h.s. becomes *proportional* to the first order variation $y_t^{(1)}$ in n^{th} power in the r.h.s. (for $j_i = 1$). Therefore the behaviours are comparable $\sqrt[n]{y_t^{(n)}} \sim y_t^{(1)}$.

If we introduce a homogeneous with respect to this symmetry expression

$$\rho_n(y,t) = \sum_{j=1,\ldots,n} p_j(y_t^x)\,||y_t^{(j)}||^{m/j}$$

and impose some reasonable conditions on the behaviour of nonlinearity F and hierarchy of weights p_j, related with the behaviour of nonlinearity F, one comes to the quasi-contractive estimate, e.g. [1,2].

$$\exists K \quad \rho_n(y,t) \le e^{Kt}\rho_n(y,0),$$

i.e. **essentially nonlinear** (different powers m/j) **a priori estimate on the regularity** of solution y_t^x with respect to the initial data x (recall that $y_t^{(n)} = \partial_x^{(n)} y_t^x$). Because in the very basis of nonlinear estimates lie monotone methods of nonlinear analysis, such estimates become applicable even in the infinite dimensions.

We also remark, that in the case of globally Lipschitz coefficients with bounded derivatives weights $p_j = 1$. This leads to a qualitively different understanding of the Cauchy–Liouville–Picard regularity scheme, that does not ground on the implicit function techniques.

Turning to our model of stochastic equation on manifold (2), one can also hope to get similar a priori estimates on regularity. Here, however, arises a new fundamental problem: how to define the high-order variations in a geometrically correct way.

3   Generalization of Riemannian covariant derivative or invariance of variations with respect to (y_t^x) coordinate

Today the procedure of the geometrically correct construction of differential flows on manifolds is quite clear. Adding the stochastic terms does not change the picture: in spite of the additional terms, arising in Ito formula, the stochastics could be agreed with the geometry, e.g. via Stratonovich differentials, special second order bundles or orthoframe liftings of diffusions [4–10].

However the further problem of consistency with problematics of **differential geometry**, namely *how the geometrically invariant differentials are constructed from geometrically invariant objects*, in fact, was not considered yet. The attempts to consider variations as defined via derivatives in directions of vector fields, covariant Riemannian derivatives lose an important property of geometric invariance with respect to the process y_t^x.

Let us explain what is meant. Suppose that some process on manifold y_t (of diffusion or any other nature) travels over manifold and enters some vicinity $U \subset M$ with coordinate functions $\varphi = (\varphi^i)_{i=1}^{\dim M}$, $\varphi : U \to \mathbb{R}^{\dim M}$. Then one can speak about the coordinates of process $y_t^i = (\varphi^i) \circ y_t$ when it stays in this vicinity.

Let $\mathcal{D}$ be some first order differentiation operation, correctly defined on diffusion process y_t. It could be of any nature, like ∂_x or stochastic derivative with respect to the random parameter, the principal moment is that the first order differentiation obeys chain rule

$$\mathcal{D}(f \circ y) = (f' \circ y)\,\mathcal{D}y.$$

Because the local coordinate changes $y^{i'} = \varphi^{i'}(y_t) = (\varphi^{i'} \circ \varphi^{\text{inv}})(y^i)_{i=1}^{\dim M}$ represent a particular case of locally defined functions, one gets rule

$$\mathcal{D}y^{m'} = \frac{\partial y^{m'}}{\partial y^m}\mathcal{D}y^m.$$

Therefore *the expression $\mathcal{D}y$ becomes a **vector field** with respect to the "coordinate" changes $(y) \to (y')$ of diffusion "variable" y*, though, of course, the diffusion process y_t does not determine some coordinate system, like local coordinate mappings φ, φ' do.

By similar to the classical differential geometry arguments, related with the construction of covariant derivatives, *there is no other way to define correctly the high-order derivatives $(\widetilde{\mathcal{D}})^i y$, but introduce additional terms with connection $\Gamma(y_t)$*.

The correct recurrent definition of the invariant high-order derivative then becomes

$$\widetilde{\mathcal{D}}y^m = \mathcal{D}y^m, \qquad \widetilde{\mathcal{D}}[\,(\widetilde{\mathcal{D}})^i y^m\,] = \mathcal{D}[\,(\widetilde{\mathcal{D}})^i y\,] + \Gamma_{p\,q}^{\ m}(y)\,[\,(\widetilde{\mathcal{D}})^i y^p\,]\mathcal{D}y^q,$$

here must also arise factors $\mathcal{D}y^q$ in the term with $\Gamma(y_t)$.

These additional terms in definition of higher-order derivatives $\widetilde{\mathcal{D}}^n$ guarantee the preservance of vector transformation law with respect to the $(y) \to (y')$ coordinate transformations:

$$(\widetilde{\mathcal{D}})^n y^{m'} = \frac{\partial y^{m'}}{\partial y^m}(\widetilde{\mathcal{D}})^n y^m, \qquad \forall\, n \geq 1.$$

Such $(y) \to (y')$ invariance, or, if one returns to the very beginning, the invariance with respect to changes of local coordinates $(y) \to (y')$ in vicinity, where travels y_t, represents a new purely geometric requirement, which for a long time remained in the shadow.

As a final remark let us also observe that the way to introduce the new type derivatives is independent on particular approach we choose to define the differential flow on manifold, actually **it works for any differential equations (high-order, etc) on manifolds**, because, by consideration above, symbol $y \in M$ must have values in manifold, but nothing more.

Now we can turn to the correct construction of high-order variations of process y_t^x (2) and develop nonlinear estimates on variations to this setting.

4 High order variations on initial data. Tensors on (x) and (y_t^x) coordinates

Consider the first order variation $\frac{\partial (y_t^x)^m}{\partial x^k}$, that represents a vector field on index m for $(y) \to (y')$ "coordinate" transformations and covector field on index k for $(x) \to (x')$ coordinate changes.

From arguments above we can immediately conclude that the definition of *geometrically invariant high-order variations* must include terms with $\Gamma(x)$ and $\Gamma(y)$ to guarantee the preservance of tensorial character on both image $(y) \to (y')$ and domain $(x) \to (x')$ coordinate changes of mapping $x \to y_t^x$.

Definition 1. Variations of differential flow y_t^x with respect to the initial data are defined recurrently by

$\mathbf{1^{st}}$ **order.** $\mathbb{W}_k^x y^m = \dfrac{\partial (y_t^x)^m}{\partial x^k}$ vector on m in (y),
                                           covector on k in (x).

High order. For $\gamma = (j_1, \ldots, j_n)$ the high-order variation $\mathbb{W}_\gamma^x y^m = \mathbb{W}_{j_n}^x \cdots \mathbb{W}_{j_1}^x y^m$ is defined by

$$\mathbb{W}_k^x(\mathbb{W}_\gamma^x y^m) = \nabla_k^x(\mathbb{W}_\gamma^x y^m) + \Gamma_{p\,q}^{\ m}(y_t^x)\mathbb{W}_\gamma^x y^p \frac{\partial y^q}{\partial x^k}$$

$$= \partial_k^x(\mathbb{W}_\gamma^x y^m) - \underbrace{\sum_{j \in \gamma} \Gamma_{k\,j}^{\ h}(x)\mathbb{W}_{\gamma|_{j=h}}^x y^m}_{\textbf{old covariant derivative}} + \underbrace{\Gamma_{p\,q}^{\ m}(y_t^x)\mathbb{W}_\gamma^x y^p \frac{\partial y^q}{\partial x^k}}_{\textbf{new term}}. \tag{5}$$

From the point of view of classical Riemannian geometry such definition of the high-order invariant variation of y with terms $\Gamma(x)$, $\Gamma(y)$ and $\frac{\partial y_t^x}{\partial x}$ provides generalization of the classical covariant Riemannian derivative. Unlike all already existing torsion, polynomial connection and other generalizations of variation, defined primarily at point x, it depends not only on initial point of differentiation x, but also on behaviour of process at point y.

The above definition naturally generalizes to the tensors on both (x) and (y_t^x) coordinates: object $u_{(j/\beta)}^{(i/\alpha)}$ represents a tensor on (x) and (y_t^x) coordinates iff it is $T_x^{p,q}M$ tensor on multi-indexes $(i) = (i_1, \ldots, i_p)$, $(j) = (j_1, \ldots, j_q)$ with respect to the local coordinates (x^k) and $T^{r,s}M$ tensor on multi-indexes $(\alpha), (\beta)$ with respect to the local coordinates (y^m). In other words, after the simultaneous change of local coordinate systems $(x^k) \to (x^{k'})$ and $(y^m) \to (y^{m'})$ one has transformation law

$$u_{(j/\beta)}^{(i/\alpha)} = \frac{\partial x^{(i)}}{\partial x^{(i')}} \frac{\partial x^{(j')}}{\partial x^{(j)}} \frac{\partial y^{(\alpha)}}{\partial y^{(\alpha')}} \frac{\partial y^{(\beta')}}{\partial y^{(\beta)}} u_{(j'/\beta')}^{(i'/\alpha')}$$

with Jacobians $\frac{\partial x^{(i)}}{\partial x^{(i')}} = \frac{\partial x^{i_1}}{\partial x i'_1} \cdots \frac{\partial x^{i_p}}{\partial x^{i'_p}}$. Simple examples of such tensors provide already introduced high-order variations or superpositions with usual tensors $u^{(i)}_{(j)}(x) v^{(\alpha)}_{(\beta)}(y^x_t)$.

The invariant (x)-derivative of tensor is defined in a similar to (5) way

$$\mathbb{W}^x_k u^{(i/\alpha)}_{(j/\beta)} = \frac{\partial}{\partial x^k} u^{(i/\alpha)}_{(j/\beta)} + \sum_{s\in(i)} \Gamma^{\ s}_{k\ h}(x) u^{(i/\alpha)|_{s=h}}_{(j/\beta)} - \sum_{s\in(j)} \Gamma^{\ h}_{k\ s}(x) u^{(i/\alpha)}_{(j/\beta)|_{s=h}}$$

$$+ \sum_{\rho\in(\alpha)} \Gamma^{\ \rho}_{\gamma\ \delta}(y^x_t) u^{(i/\alpha)|_{\rho=\delta}}_{(j/\beta)} \frac{\partial y^\delta}{\partial x^k} - \sum_{\rho\in(\beta)} \Gamma^{\ \gamma}_{\rho\ \delta}(y^x_t) u^{(i/\alpha)}_{(j/\beta)|_{\rho=\gamma}} \frac{\partial y^\delta}{\partial x^k}.$$

First line corresponds to the covariant derivative on (x^k) coordinates, second line makes the resulting expression to be tensor with respect to the coordinates (y^m). Next lemma checks the tensorial transformation property for the (x) covariant derivative.

Theorem 1. *The covariant (x) derivative defines a tensor of higher valence, i.e. transformation law holds*

$$\mathbb{W}_k u^{(i/\alpha)}_{(j/\beta)} = \frac{\partial x^{k'}}{\partial x^k} \frac{\partial x^{(i)}}{\partial x^{(i')}} \frac{\partial x^{(j')}}{\partial x^{(j)}} \frac{\partial y^{(\alpha)}}{\partial y^{(\alpha')}} \frac{\partial y^{(\beta')}}{\partial y^{(\beta)}} \mathbb{W}_{k'} u^{(i'/\alpha')}_{(j'/\beta')}.$$

Before turning directly to the proof, let us briefly recall the definition of connection. Connection form is a mapping

$$\text{Hess} : C^\infty(M) \to C^\infty(T^{0,2}M)$$

that fulfills property

$$\text{Hess}\,(h\circ f) = \partial_j h \circ f\,\text{Hess}\,f^j + \partial_{ij}h\circ f\,df^i \otimes df^j.$$

Its components, known as Cristoffel symbols, are defined by taking Hessians of local coordinates

$$\text{Hess}\,x^i = -\Gamma^{\ i}_{j\ k} dx^i \otimes dx^j.$$

In particular, for any function, viewed as a function of local coordinates we have

$$\text{Hess}\,(f\circ x) = (\partial_{ij}f - \Gamma^{\ k}_{i\ j}\partial_k f)dx^i \otimes dx^j.$$

Proof. One should write the expression of (x) derivative and substitute the transformation law of tensor on (x) and (y) coordinates to get

$$\mathbb{W}_k u^{(i/\alpha)}_{(j/\beta)} = [\text{Jacobians}]\frac{\partial x^{k'}}{\partial x^k} \partial_{k'} u^{(i'/\alpha')}_{(j'/\beta')}$$

$$+ \sum_{s\in(i)} [\text{Jacobians with } s\text{-out}]\left\{ \frac{\partial^2 x^s}{\partial x^k \partial x^{s'}} u^{(i'/\alpha')}_{(j'/\beta')} + \Gamma^{\ s}_{k\ h}(x) \frac{\partial x^h}{\partial x^{h'}} u^{(i'/\alpha')|_{s'=h'}}_{(j'/\beta')} \right\} \qquad (A)$$

$$+ \sum_{\rho\in(\alpha)} [\text{Jacobians with } \rho\text{-out}]\left\{ \frac{\partial^2 y^\rho}{\partial x^k \partial y^{\rho'}} u^{(i'/\alpha')}_{(j'/\beta')} + \Gamma^{\ \rho}_{\gamma\ \delta}(y) \frac{\partial y^\gamma}{\partial y^{\gamma'}} u^{(i'/\alpha')|_{\rho'=\gamma'}}_{(j'/\beta')} \frac{\partial y^\delta(x)}{\partial x^k} \right\} \qquad (B)$$

$$- \sum_{s\in(j)} (\text{similar to (A)}) - \sum_{\rho\in(\beta)} (\text{similar to (B)}), \qquad (C)$$

where [Jacobians] means the Jacobians from the transformation law of $u_{(j/\beta)}^{(i/\alpha)}$ and [Jacobians with s-out] means that in product the corresponding Jacobian with index s is omitted. Line (C) is written in analogue to the lines (A)–(B) with corresponding modifications.

Transformation of line (A). By definition of Cristoffel symbols

$$\text{Hess}\,(x^1) = -\Gamma_2{}^1{}_3 dx^2 \otimes dx^3 = -\Gamma_2{}^1{}_3 \frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{3'}} dx^{2'} \otimes dx^{3'}.$$

From another side from definition of connection

$$\text{Hess}\,x^1(x') = \frac{\partial x^1}{\partial x^{1'}}\text{Hess}\,x^{1'} + \frac{\partial^2 x^1}{\partial x^{2'}\partial x^{3'}}dx^{2'} \otimes dx^{3'} = \left[-\frac{\partial x^1}{\partial x^{1'}}\Gamma_{2'}{}^{1'}{}_{3'} + \frac{\partial^2 x^1}{\partial x^{2'}\partial x^{3'}} \right] dx^{2'} \otimes dx^{3'}$$

that leads to relation

$$\frac{\partial^2 x^1}{\partial x^2 \partial x^{3'}} = \frac{\partial x^{2'}}{\partial x^2}\frac{\partial^2 x^1}{\partial x^{2'}\partial x^{3'}} = \Gamma_{2'}{}^{1'}{}_{3'}(x')\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^{2'}}{\partial x^2} - \Gamma_2{}^1{}_3\frac{\partial x^3}{\partial x^{3'}} \tag{D}$$

and one can transform each term in (A) to the form

$$(A) = [\text{Jacobians with } s\text{-out}]\,\frac{\partial x^s}{\partial x^{s'}}\frac{\partial x^{k'}}{\partial x^k}\Gamma_{k'}{}^{s'}{}_{h'}(x')u_{(j'/\beta')}^{(i'/\alpha')|_{s'=h'}}. \tag{E}$$

Transformation of line (B). Changing the coordinates $(x) \leftrightarrow (y)$ in (D) we get

$$\frac{\partial^2 y^1}{\partial y^2 \partial y^{3'}} = \Gamma_{2'}{}^{1'}{}_{3'}(y')\frac{\partial y^1}{\partial y^{1'}}\frac{\partial y^{2'}}{\partial y^2} - \Gamma_2{}^1{}_3(y)\frac{\partial y^3}{\partial y^{3'}}.$$

This leads to

$$\frac{\partial^2 y^1}{\partial x^k \partial y^{3'}} = \frac{\partial y^2}{\partial x^k}\frac{\partial^2 y^1}{\partial y^2 \partial y^{3'}} = \Gamma_{2'}{}^{1'}{}_{3'}(y')\frac{\partial y^1}{\partial y^{1'}}\frac{\partial y^{2'}}{\partial x^{k'}}\frac{\partial x^{k'}}{\partial x^k} - \Gamma_2{}^1{}_3(y)\frac{\partial y^3}{\partial y^{3'}}\frac{\partial y^2}{\partial x^k}$$

and permits to transform terms in (B) to the form

$$(B) = [\text{Jacobians with } \rho\text{-out}]\,\frac{\partial y^\rho}{\partial y^{\rho'}}\frac{\partial x^{k'}}{\partial x^k}\Gamma_{\gamma'}{}^{\rho'}{}_{\delta'}\frac{\partial y^{\delta'}}{\partial x^{k'}}u_{(j'/\beta')}^{(i'/\alpha')|_{\rho'=\gamma'}}. \tag{F}$$

Finally, collecting together terms (E), (F) and transforming the remaining terms in (C) we come to the (x) covariant derivative in new coordinate systems

$$\mathbb{W}_k u_{(j/\beta)}^{(i/\alpha)} = [\text{Jacobians}]\frac{\partial x^{k'}}{\partial x^k}\mathbb{W}_{k'} u_{j'/\beta')}^{(i'/\alpha')}. \qquad\qquad\blacksquare$$

An important property of new (x) covariant derivative is a chain rule, which does not hold for traditional covariant derivative.

Corollary 1. *Let $u_{(\beta)}^{(\alpha)}$ be a tensor on manifold M. Then for superposition one has chain rule*

$$\mathbb{W}_k u_{(\beta)}^{(\alpha)}(y(x)) = (\nabla_\ell u_{(\beta)}^{(\alpha)})(y(x))\frac{\partial y^\ell}{\partial x^k}.$$

Proof. Using definition of (x) derivative we get

$$\mathbb{W}_k u_{(\beta)}^{(\alpha)}(y(x)) = \partial_k^x u_{(0/\beta)}^{(0/\alpha)}(y(x)) + \sum_{\rho \in (\alpha)}\Gamma_\gamma{}^\rho{}_\delta(y)u_{(0/\beta)}^{(0/\alpha)|_{s=\gamma}}(y)\frac{\partial y^\delta}{\partial x^k} - \sum_{\rho \in (\beta)}\Gamma_\rho{}^\gamma{}_\delta(y)u_{(0/\beta)|_{\rho=\gamma}}^{(0/\alpha)}\frac{\partial y^\delta}{\partial x^k}.$$

From chain rule for ∂_k^x and definition of usual covariant derivative follows the statement. $\qquad\blacksquare$

5   Role of curvature in the regularity problems

Having in hands the correct procedure of differentiation of tensors of y, such as defined by coefficients $A_\bullet(y_t^x)$ of equation (2), we can find the corresponding variational equations. Taking the partial derivative of (2) we find

$$\delta\left(\frac{\partial y^m}{\partial x^k}\right) = \left(\frac{\partial}{\partial x^k}A_0^m(y)\right)dt + \left(\frac{\partial}{\partial x^k}A_\alpha^m(y)\right)\delta W^\alpha.$$

By adding and subtracting the terms with $\Gamma(y)$ to single out the covariant (x)-derivative of vector fields $A_0(y)$, $A_\alpha(y)$ on image coordinates (y) we have

$$\delta\left(\frac{\partial y^m}{\partial x^k}\right) = \left(\mathbb{W}_k^x A_0^m(y) - \Gamma_{p\,q}^{\;m}(y)A_0^p\frac{\partial y^q}{\partial x^k}\right)dt + \left(\mathbb{W}_k^x A_\alpha^m(y) - \Gamma_{p\,q}^{\;m}(y)A_\alpha^p\frac{\partial y^q}{\partial x^k}\right)\delta W^\alpha.$$

Noting that the terms near connection contain the differential of process y we finally get invariant form of equation on the first variation

$$\delta\left(\frac{\partial y^m}{\partial x^k}\right) = -\Gamma_{p\,q}^{\;m}(y)\frac{\partial y^p}{\partial x^k}\delta y^q + \mathbb{W}_k^x(A_\alpha^m(y))\delta W^\alpha + \mathbb{W}_k^x(A_0^m(y))dt. \tag{6}$$

Therefore up to the parallel transition term with $\Gamma(y)$ the increments of first order variation are determined by covariant (x)-derivatives of coefficients. We take this observation as the recurrence base in the search for high-order variational equations.

Theorem 2. *Suppose that the equation on variation $\mathbb{W}_\gamma^x y^m, |\gamma| \geq 1$ is written in form*

$$\delta(\mathbb{W}_\gamma^x y^m) = -\Gamma_{p\,q}^{\;m}(\mathbb{W}_\gamma^x y^p)\delta y^q + M_{\gamma\;i}^{\;m}\delta W^i + N_\gamma^m dt. \tag{7}$$

Then the next order variation $\mathbb{W}_k^x \mathbb{W}_\gamma^x y^m = \mathbb{W}_{\gamma\cup\{k\}}^x y^m$ fulfills relation

$$\delta(\mathbb{W}_{\gamma\cup\{k\}}^x y^m) = -\Gamma_{p\,q}^{\;m}(\mathbb{W}_{\gamma\cup\{k\}}^x y^p)\delta y^q + R_{m\;\ell q}^{\;p}(\mathbb{W}_\gamma^x y^p)\frac{\partial y^\ell}{\partial x^k}\delta y^q$$
$$+ (\mathbb{W}_k^x M_{\gamma\;i}^{\;m})\delta W^i + (\mathbb{W}_k^x N_\gamma^m)dt. \tag{8}$$

Therefore the coefficients of variational equations are recurrently related by

$$M_{\gamma\cup\{k\}\;i}^{\;m} = \mathbb{W}_k^x M_{\gamma\;i}^{\;m} + R_{m\;\ell q}^{\;p}(\mathbb{W}_\gamma^x y^p)\frac{\partial y^\ell}{\partial x^k}A_i^q, \qquad N_{\gamma\cup\{k\}}^m = \mathbb{W}_k^x N_\gamma^m + R_{m\;\ell q}^{\;p}(\mathbb{W}_\gamma^x y^p)\frac{\partial y^\ell}{\partial x^k}A_0^q.$$

Above R denotes curvature tensor.

Remark 1. Introduction of new type variations finally let us find a place of curvature (8) in the regularity properties. Having started from variations, defined by derivatives in directions of vector fields or by covariant Riemannian derivatives, we would get a set of (y_t^x) non-invariant equations with exploding number of non-invariant coefficients, e.g. [4, 6]. In other words, additional terms in (5) compactificate the variational equations to a simplest possible form (8), that provides one more argument in favor of introduction of new type variations.

6   Nonlinear estimates on regularity of differential flows on manifolds. Applications to the semigroup properties

From structure of variational equations (8) and chain property of covariant (x)-derivative $\mathbb{W}_k^x u_{(j)}^{(i)}(y_t^x) = [\nabla_\ell u_{(j)}^{(i)}](y_t^x)\frac{\partial y^\ell}{\partial x^k}$ we see that symmetry $\mathbb{W}^x y_t^x \sim \sqrt[n]{(\mathbb{W}^x)^n y_t^x}$ manifests again, leading to the related nonlinear expression

$$\rho_n(y,t) = \sum_{j=1,...,n} \mathbf{E}\, p_j(\rho^2(y_t^x, z))\,||(\mathbb{W}^x)^j y_t^x||^{m/j}$$

and nonlinear estimate on regularity. Above z is some fixed point of manifold.

Theorem 3. *Introduce notation* $\widetilde{A_0} = A_0 + \sum_\alpha \nabla_{A_\alpha} A_\alpha$. *Suppose conditions hold:*

dissipativity and differential coercitivity:

$$\forall\, C \,\exists\, K_C \,\forall\, x \in M \quad \langle \widetilde{A_0}(x), \nabla^x \rho^2(x,z) \rangle + C \sum_\alpha \|A_\alpha(x)\|^2 \le K_C(1 + \rho^2(x,z)),$$

$$\forall\, C, C' \,\exists\, K_C \quad \langle \nabla \widetilde{A_0}[h], h \rangle + C \sum_{\alpha=1}^d |\nabla A_\alpha[h]|^2 + C' \sum_\alpha \langle R(A_\alpha, h) A_\alpha, h \rangle \le K_C \|h\|^2.$$

nonlinear behavior of all derivatives of curvature and coefficients:

$$\exists\, \mathbf{k} \,\forall\, j \quad \|(\nabla)^j \widetilde{A_\bullet}(x), R(x)\| \le (1 + \rho(x,z))^{\mathbf{k}}.$$

Then for polynomial weights $p_j \ge 1$, which fulfill hierarchy

$$\forall\, j_1 + \cdots + j_s = i \quad [p_i(z)]^i (1 + |z|)^{m\mathbf{k}} \le [p_{j_1}(z)]^{j_1} \cdots [p_{j_s}(z)]^{j_s}$$

the **nonlinear estimate on regularity** *holds*

$$\exists\, K \quad \rho_n(y, t) \le e^{Kt} \rho_n(y, 0).$$

Proof proceeds similar to [1] with corresponding geometric complications, it will appear in [3].

Turning to the study of regularity properties of semigroup $(P_t f)(x) = \mathbf{E}\, f(y_t^x)$ one can prove representation, that relate the behaviour of covariant derivatives of function and its evolution

$$(\nabla^x)^n P_t f(x) = \sum_{j_1 + \cdots + j_s = n,\ s=1,\ldots,n} \mathbf{E}\, \langle (\nabla^y)^s f(y_t^x), (\mathbb{W}^x)^{j_1} y_t^x \otimes \cdots \otimes (\mathbb{W}^x)^{j_s} y_t^x \rangle.$$

In particular, we get next argument in favor of new type variations: they relate classical covariant derivatives of function and its evolution – and only they!

Because in the above representation new type variations $(\mathbb{W}^x)^j y_t^x$ appear as convolutional kernels, one can get the regular properties of semigroup from nonlinear estimates on regularity. This is a subject of [3].

Acknowledgements

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