



The Quark-Hadron Duality

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I review the notion of the quark-hadron duality from the modern perspective. The proper theoretical framework in which the problem can be formulated and treated is Wilson's operator product expansion (OPE). Two models developed for the description of duality violations are considered in some detail: one is instanton-based, another resonance-based. The mechanisms they represent are complementary.

Prelude

Curiously, this is the tenth talk I give in Italy, since I came here for the first time in the late 1980's. The Soviet Empire was still in existence, albeit its ideological foundation — “scientific communism”¹ — was already tumbling. It was not just Italy, but a charmed corner — Capri, the land of fairy tales — where, by a perverted twist of fortune, I found myself in 1988 with Vladimir Naumovich Gribov.

This magic word, Capri, was familiar to every Soviet schoolboy. Our school lessons of Russian literature were ideologically charged. We spent much time studying the heritage of the “greatest proletarian writer of all times,” Maksim Gorky. In 1906 Gorky settled in Capri where other proletarian leaders, such as Lenin, used to come for a vacation. He spent many years in Capri, with intermissions, until in 1931 he was finally lured by Stalin back to the USSR, “worker's paradise.” Apparently, human feelings were not alien to the proletarian writer, despite his preoccupation with the socialist idea. He fell in love with Capri and Italy at large; as a result of this love affair twenty seven *Tales of Italy* appeared in 1906 – 1917 presenting a strange mixture of socialist propaganda with stories of larger-than-life human passions and colorful sketches of Capri, the Bay of Naples, Sorrento and other romantic places, drenched in golden sunlight, with craggy cliffs and sweet scent of flowers wafting from the hills. He cherished the breath of antiquity which blurred the distinction between the past and the present.

I do not know whether it was in the curriculum, or our literature teacher was just overzealous, but we had to learn a couple of Gorky's tales by heart, and some passages produced such a strong impression on me that I remembered them once and for all.

¹This combination of words is, of course, oxymoronic, but that's the way I learned it at school.



“... The sun melts in the blue midday sky, pouring its hot rainbow-hued rays on to sea and earth. The drowsy sea exhales an opalescent mist, the blue water gleams like steel, and a strong scent of brine is wafted ashore. The waves plash lazily against the grey boulders, spill over their backs and on to the whispering pebbles; they are small waves, as transparent as glass and untouched by foam. A purple haze enwraps the mountain, the grey olive leaves are like old silver in the sunlight, the dark velvety green of the gardens terracing the hills is lit up by the golden glow of lemons and oranges, the scarlet pomegranate blossoms smile their vivid smile and there are flowers everywhere...”

“... The rocky coast descends in jagged ledges down to the sea; it is covered with a luxurious tangle of dark vines, lemons and figs... Flowers, gold, red and white, smile gently through the dense foliage dropping steeply down to the sea, and the yellow and orange fruits remind one of stars on a warm moonlight night when the sky is dark and the air is moist...”

“... The cicadas are strumming. It is as if thousands of metal strings were stretched taut among the thick foliage of the olive trees, the wind stirs the brittle leaves, they touch the strings and this light, ceaseless contact fills the air with intoxicating sound. It is not exactly

music, yet it seems as if invisible hands were tuning hundreds of invisible harps, and one waits in tense expectancy for the tuning to cease, and for a grand string orchestra to strike up a triumphant hymn to the sun, sky and sea...”

“... A bluish mist heavy with the sweet scent of the golden furze floats skyward from the rocky island. Standing in the midst of the dark mass of sleepy water under the pale cupola of the sky, the island is like an altar to the Sun God...”

I kept thinking: “was it on *my* planet? and if Gorky, the proletarian writer, could see this, why couldn’t I?” The surrounding reality and my everyday experiences taught me that that was a poetical exaggeration. And even if there was some truth in this description, I would never ever become a part of it. So, it was better to forget ... but I couldn’t.

My journey to Italy took thirty years. When I finally descended on Capri for the first time in 1988 I instantly realized there was no exaggeration, it was all real — sweetly intoxicating scent of flowers, melting sun, melodic Italian speech (I did not understand a word, but I heard in them the voice of a violin, and, sometimes, a passionate cello), washed-out jagged rocks, dark-eyed curly-haired Italian women, with mystery in their eyes, undoubtedly the most beautiful in the world...

I came to Capri for the second time in 1994; in a sense this encounter brought even more excitement in my life. This is a separate story, however.

1 Duality

It is time to interrupt my recollections and turn to physics which can be no less captivating. Dictionaries define *duality* as something with seemingly contradictory qualities which are in fact manifestations of one and the same phenomenon seen from complementary angles. With Seiberg’s duality, Seiberg-Witten’s duality, string dualities, and so on, it is obvious that our age is the age of duality. I will discuss today a particular topic going under the name of *quark-hadron duality* — the one whose understanding is most needed by our experimental colleagues. The notion of the quark-hadron duality exists in high-energy physics for 30 years or so. Apparently, artistic imagination discovered it much earlier. The above statement is illustrated by Fig. 1 exhibiting a fragment of Escher’s “Sky and Water.” Here the gaps between the birds are fishes, while the gaps between the fishes are birds.

The last time I delved in this problem in earnest was in 2000 when I was working on *Handbook of QCD*



Figure 1. An illustration of duality: a fragment of Escher’s “Sky and Water.”

[1]. As I have mentioned, this aspect of QCD is extremely important in a wide range of practical applications — e^+e^- annihilation, deep inelastic scattering, jet physics, τ decays, inclusive heavy flavor decays, to name just a few. At the same time it is rarely discussed by theorists; bits and pieces that you can find in textbooks reflect the understanding of this subject of twenty years ago, at best. A considerable progress was made in the understanding of the quark-hadron duality in the 1990’s [2–10]. But then it again fell in neglect. The organizers of this Workshop invited me to review the subject. At first I hesitated whether to accept it, due to the lack of progress in the last three years. But then I decided that *QCD@Work 2003*, (with the emphasis on “Work,” right?) provides an excellent forum for resurrecting this topic. My task is to get you involved, so that you could develop the subject further.

2 Early inception and a long latent period

Quantum chromodynamics (QCD) is a very strange theory. All theoretical calculations are done in terms of quarks and gluons. At the same time, quarks and gluons are never detected experimentally. What is actually produced and detected in experimental devices are hadrons: pions, kaons, protons, *etc.* The quark-hadron duality allows one, under certain circumstances, to bridge the gap between the theoretical predictions and experimentally observable quantities. The idea was first formulated at the dawn of the QCD era by Poggio, Quinn and Weinberg [10], who suggested that certain inclusive hadronic cross sections at high energies, being appropriately averaged over an energy range, had to (approximately) coincide with the cross sections one could calculate in the quark-gluon perturbation theory. In spite of the dramatic developments in QCD in the subsequent two decades the notion of the quark-hadron duality remained at the 1976 level, with the very basic questions unanswered. These questions are:

- What energy is considered to be high enough for the

quark-hadron duality to set in, and what accuracy is to be expected?

- What weight function is appropriate for the averaging of the experimental cross sections?
- If the theoretical prediction includes only perturbation theory, should one limit oneself to some particular order in the α_s series?
- Do we have to include known nonperturbative effects (e.g. condensates) in the theoretical prediction?
- Given a definition of the quark-hadron duality, can one estimate deviations from duality and how?

Of course, not all of these questions are independent. For instance, answering the last question, one will simultaneously learn the boundary energy.

Systematic explorations of these and related issues started in earnest [2] in 1994. As in many other instances, this was dictated by practical needs. Previously, the accuracy of the experimental data on hard inclusive processes was rather modest, so that the Poggio-Quinn-Weinberg prescription was good enough. By 1994 the data, mostly associated with the b quark physics, became so precise and the questions raised so pressing, that a much better theoretical understanding became imperative.

In the problem of quark-hadron duality one cannot expect help from lattices. The duality violation is a phenomenon inseparable from the Minkowskian kinematics; numerical Euclidean approaches, such as lattice QCD, have nothing to say on this issue. Analytic methods are needed.

I will discuss the modern formulation of the problem based on Wilson’s operator product expansion (OPE). Then I will review models which were designed to give us a certain idea of (and a degree of control over) the deviations from duality. There are two classes of such models: instanton-based and resonance-based. They are complementary to each other. Finally, I will present sample applications in the processes of the current interest, such as the hadronic τ decays.

3 The quark-hadron duality: what does that mean?

Let us consider an idealized theory — QCD, with two massless quarks, u and d . We are interested in the total hadronic cross section of the e^+e^- annihilation. The “photon” in our theory is idealized too, its coupling to the quark current has the form

$$J_\mu = \frac{\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d}{\sqrt{2}}. \quad (1)$$

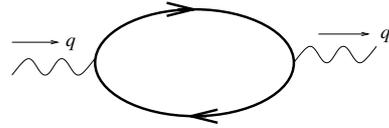


Figure 2. The one-loop graph determining the polarization operator and the spectral density in the leading (parton) approximation. The “photon” momentum is denoted by q .

(In fact, this is the isovector part of the actual electromagnetic current). We define the two-point function $\Pi_{\mu\nu}$

$$\Pi_{\mu\nu} = i \int e^{iqx} d^4x \langle 0 | T \{ J_\mu(x) J_\nu^\dagger(0) \} | 0 \rangle. \quad (2)$$

Here q is the total momentum of the quark-antiquark pair. Due to the current conservation $\Pi_{\mu\nu}$ is transversal,

$$\Pi_{\mu\nu} = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(q^2). \quad (3)$$

The experimentally observable quantity is the imaginary part of $\Pi(q^2)$ at positive values of q^2 (i.e. above the physical threshold of the hadron production), the spectral density,

$$\rho(s) = \frac{12\pi}{N_c} \text{Im} \Pi(s), \quad s \equiv q^2. \quad (4)$$

Up to a normalization, the expression above coincides with the cross section of e^+e^- annihilation into hadrons measured in the units $\sigma(e^+e^- \rightarrow \mu^+\mu^-)$, the famous ratio R .

Theoretically one can calculate $\Pi(q^2)$ in the deep Euclidean domain, at negative q^2 . For instance, from the free quark loop of Fig. 2 one gets

$$\Pi(Q^2) \rightarrow -\frac{N_c}{12\pi^2} \ln Q^2, \quad Q^2 \equiv -q^2. \quad (5)$$

Performing the analytic continuation to the Minkowski domain and taking the imaginary part one arrives at

$$\rho(s)_{\text{theor}} \rightarrow 1, \quad s \rightarrow \infty. \quad (6)$$

This is the spectral density in the theory with free quarks, i.e. $\alpha_s = 0$. There are various corrections to the free quark result (5). The perturbative gluon exchanges give rise to the $\alpha_s(Q^2)$ series. Nonperturbative (power) corrections come from the quark and gluon condensates and from other sources, e.g. the small-size instantons. A systematic method of handling the theoretical calculations of $\Pi(Q^2)$ in the *deep Euclidean* domain is provided by Wilson’s operator product expansion (OPE) [11]. Essentially, this is a bookkeeping procedure: one consistently separates the

short-distance contributions (i.e. those coming from distances $\leq \mu^{-1}$) from the large distance contributions (i.e. those coming from distances $\geq \mu^{-1}$). Here μ is a theoretical parameter (usually referred to as the normalization point) separating the two domains. The choice of μ is a matter of convenience — observable quantities do not depend on it.

The short-distance contributions determine the coefficients $C_n(q)$ of OPE,

$$\begin{aligned} D(q^2) &= \sum_n C_n(q; \mu) \langle \mathcal{O}_n(\mu) \rangle, \\ D(Q^2) &\equiv -(4\pi^2) Q^2 (d\Pi/dQ^2). \end{aligned} \quad (7)$$

The normalization point μ is indicated explicitly. The sum in Eq. (7) runs over all possible Lorentz and gauge invariant local operators built from the gluon and quark fields. The operator of the lowest (zero) dimension is the unit operator \mathbf{I} , followed by the gluon condensate $G_{\mu\nu}^2$, of dimension four. The four-quark condensate gives an example of dimension-six operators.

At short distances QCD is well described by the quark-gluon perturbation theory. Therefore, as a first approximation, it is reasonable to calculate C_n perturbatively, in the form of expansion in $\alpha_s(Q) \sim (\ln Q)^{-1}$. This certainly does not mean that the coefficients C_n are free from nonperturbative (nonlogarithmic) terms. The latter may and do appear in C_n 's; they are of the type $\sim Q^{-\gamma}$ where γ is a positive number, not necessarily an integer. Such terms in C_n 's are generated, for instance, by the small-size instantons. Another source of the power terms in C_n , of a technical rather than dynamical nature, is the normalization point μ : in calculating $C_n(\mu)$ one must remove all soft contributions with off-shellness less than μ .

The condensate terms in Eq. (7) give rise to corrections of the type $(\Lambda_{\text{QCD}}/Q)^n$ where n is an integer ≥ 4 , the (normal) dimension of the operator \mathcal{O}_n , modulo logarithms associated with the anomalous dimensions. Λ_{QCD} is the scale parameter of QCD (sometimes, I will drop the subscript “QCD” for brevity) entering through the matrix elements of \mathcal{O}_n 's.

If one could calculate $\Pi(Q^2)$ in the Euclidean domain *exactly*, one could analytically continue the result to the Minkowski domain, and then take the imaginary part. The spectral density $\rho(s)_{\text{theor}}$ obtained in this way would present the *exact* theoretical prediction for the measurable hadronic cross section. *There would be no need in duality.*

In practice, our calculation of $\Pi(Q^2)$ is approximate, for many reasons. First, nobody is able to calculate the infinite $\alpha_s(Q^2)$ series for the coefficient functions, let alone the infinite condensate series. Both

have to be truncated at some finite order. A few lowest-dimension condensates that can be captured, are known approximately. The best we can do is analytically continue the *truncated* theoretical expression, term by term, from positive to negative Q^2 . For each term in the expansion the imaginary part at positive q^2 (negative Q^2) is well-defined. We assemble them together and declare the corresponding $\rho(s)_{\text{theor}}$ to be dual to the hadronic cross section $\rho(s)_{\text{exp}}$. In the given context “dual” means equal.

Let me elucidate this point in more detail. Assume that $\Pi(Q^2)$ is calculated through α_s^2 and $1/Q^4$, while the terms α_s^3 and $1/Q^6$ (with possible logarithms) are dropped. Then the theoretical quark-gluon spectral density, obtained as described above, is expected to coincide with $\rho(s)_{\text{exp}}$, with the uncertainty of order $O[(\alpha_s(s))^3]$ and $O(1/s^3)$. The uncertainty in the theoretical prediction of this order of magnitude is *natural* since terms of this order are neglected in the theoretical calculation of $\Pi(Q^2)$. If the coincidence in this corridor does take place, we say that the quark-gluon prediction is dual to the hadronic spectral density. If there are deviations going *beyond the natural uncertainty*, we call them violations of duality. Needless to say that, once our calculation of $\Pi(Q^2)$ becomes more precise, the definition of the “natural uncertainty” in $\rho(s)_{\text{theor}}$ changes accordingly.

This is the most clear-cut definition I can suggest. From the formal standpoint, it connects the duality violation issue with that of analytic continuation from the Euclidean to Minkowski domain. Negligibly small corrections (legitimately) omitted in the Euclidean calculations may and do get enhanced in Minkowski.

Before I proceed to explain, from the physical standpoint, where the violations of duality come from, and their pattern, I would like to make a remark important in the conceptual aspect. The necessity of truncation of the α_s and condensate series is not just due to our practical inability to calculate high-order terms. (For the vast majority of the theorists, myself including, “high-order” begins at the next-to-next-to-leading level.) Assume we have a “sorcerer’s stone” which would allow us to exactly calculate any term in the expansion we want. Still, we would not be able to find $\Pi(Q^2)_{\text{exact}}$ because the both series are factorially divergent. The α_s series has at least two known sources of the factorial behavior: first, the number of the Feynman graphs grows factorially at high orders [12]; second, there are graphs with renormalons [13] (of which we will not bother in what follows since the infrared renormalons — the only ones which are potentially dangerous — are totally eliminated by the introduction of the normalization point μ). The condensate series is divergent too. The factorial divergence of the condensate series is studied even to a lesser extent than

that of the α_s series. The only fact considered to be firmly established [2,3] is the divergence *per se*. It is not Borel-summable, which sets a limit on the theoretical accuracy. Including more and more terms in the series would not help, and even an optimal truncation would leave a gap between $\Pi(Q^2)_{\text{theor}}$ and $\Pi(Q^2)_{\text{exact}}$.

4 Where the duality violations come from?

Theoretical calculations of inclusive processes in QCD are performed through Wilson's OPE in the Euclidean domain. Therefore, to understand what is included in such calculations and what is left out, one should have a clear picture of limitations of OPE.

Let us return to the consideration of the correlation function of two currents,

$$\langle 0|T\{J_\mu(x)J_\nu^\dagger(0)\}|0\rangle,$$

which determines the polarization operator in Eq. (2), at negative (Euclidean) x^2 . Wilson's OPE is nothing but an expansion of $\langle 0|T\{J_\mu(x)J_\nu^\dagger(0)\}|0\rangle$ in *singularities* at $x^2 = 0$. It properly captures all terms of the type $1/x^6$ times logarithms, or $\ln x^2$, or $x^2 \ln x^2$, and so on. Every term of this type is represented in the condensate series provided that the calculation is carried out to a sufficient order.

Let me note in passing that limiting oneself to singular terms (and, in particular, to the leading singular term) is a reasonable approximation in the theories in which the asymptotic conformal regime sets in at short distances in the power-like manner. In fact, OPE in its original formulation was designed by Wilson for applications in such theories (see the first work in Ref. [11]). In QCD the approach to the asymptotically free regime is very slow — logarithmic. That's why the precision of the leading (parton) approximation is sufficient only for very rough estimates, and that's why the high-order terms, both logarithmic and power, must be kept. The time when the QCD practitioners would be satisfied by the leading, or even the first subleading approximation, is long gone.

It is clear that the function $\langle 0|T\{J_\mu(x)J_\nu^\dagger(0)\}|0\rangle$ is not fully determined by its singularities at $x^2 = 0$. Generally speaking, one may have, additionally, isolated singularities at finite x^2 , or a singularity at infinity, which are not reflected in the truncated Wilson's OPE. Consider, say, a singularity at finite x^2 of the form

$$\frac{1}{x^2 + \rho^2}. \tag{8}$$

The expansion of this function (truncated at any order) generates derivatives of $\delta(q^2)$ in the momentum

space; therefore, it has no impact on the OPE expansion at large Q^2 . This contribution is clearly missing. The Fourier transform of Eq. (8) at large Euclidean Q^2 falls off as $\exp(-Q\rho)$. Thus, this term is smaller than any of the terms in the condensate expansion. We should not forget, however, that our final goal is a prediction in the Minkowski domain. Upon analytic continuation, the exponentially small term $\exp(-Q\rho)$ loses its suppression and becomes oscillating, $\sin(-E\rho)$, where E stands for the total energy, $E = \sqrt{q^2}$. If it were not for the power suppression in the pre-exponent, such exponential/oscillating terms would be a total disaster. Since they are not seen in OPE, any prediction for the inclusive cross sections made through OPE (which is equivalent, in practice, to perturbation theory plus a few condensates) would be grossly wrong. There would be no quark-hadron duality at all. All calculations of the hard processes — from the total hadronic cross sections in e^+e^- annihilation, to jet physics, to the heavy quark decays — would be valid roughly up to a factor of two, no matter how large the energy release is.

Fortunately, one avoids the disaster due to the fact that the duality-violating exponential/oscillating terms are, in fact, suppressed as $E^{-\kappa} \sin(-E\rho)$ where κ is a positive index, which depends on the process under consideration and is typically rather large. This justifies theoretical predictions based on a few first terms in the condensate expansion. Needless to say that determining κ is one of the most important tasks in the issue of the quark-hadron duality.

Singularities at $x^2 \rightarrow \infty$ also lead to the exponential/oscillating terms of a somewhat different form, of the type $\exp(-Q^2\rho^2)$ in Euclidean. In the Minkowski domain one gets an oscillating function of the type $E^{-\eta} \sin(-E^2\rho^2)$, where η is another positive index. In one and the same process, one can expect both duality-violating components to show up. Generally speaking, the indices κ and η are unrelated; at least, at the moment we do not see any obvious relation between them. (The coincidence of κ and η in the practically important problem of the inclusive hadronic τ decays, $\kappa = \eta = 6$, seems to be accidental, see Secs. 7 and 9).

How can one isolate the singularities at $x^2 = 0$ from others diagrammatically? To answer this question it is convenient to pass to the momentum space. In the leading approximation the polarization operator $\Pi(Q^2)$ is presented by the graph of Fig. 2. The large Euclidean momentum q (remember, $Q^2 \rightarrow \infty$) flows in the photon line on the left, propagates through the fermion lines, and leaves the graph through the photon line on the right. The virtual momenta of the fermion lines typically scale as q . The first α_s correction is presented in Fig. 3. In this graph the virtual momenta

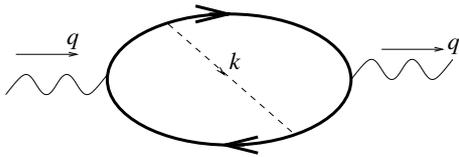


Figure 3. The one-gluon correction in the polarization operator. The gluon momentum is denoted by k .

of all lines in the loops, the fermion and the gluon, scale as q . Thus, the diagram of Fig. 3 determines the leading logarithmic correction to the parton result (5) or (6).

Although the leading contribution to the integral comes from the domain where all virtual momenta are proportional to q , there are nonvanishing contributions from other kinematical domains. For instance, one can consider a “corner” where the gluon virtual momenta k in Fig. 3 is small, $|k| < \mu$, and does not scale with q . Certainly, at small k the gluon propagator is *not* given by perturbation theory. The gluon line must be cut. This corresponds to the gluon condensate term [14] in the condensate expansion.

In general, any term in the condensate expansion can be interpreted in this way, in terms of *factorization in the momentum space*. The large external momentum q is transmitted through one or several hard lines — their momenta scale with q . This is the definition of “hardness.” The remainder of the graph is a soft part, which factors out and gives rise to gluon, quark and mixed condensates. The virtual momenta in this part of the graph do not scale with q , they are limited by a fixed parameter μ (this is the definition of “softness”).

The hard part of the graph is responsible for the coefficient functions. The fact that not all lines in the given graph are hard results in the power suppression of the corresponding coefficient function. Letting more and more lines to be soft, one obtains consecutive terms in the condensate expansion. What is missing?

It is conceivable that the number of lines through which q is transmitted becomes so large, that though $Q \rightarrow \infty$, neither of the lines is hard. Of course, in this case one cannot speak of the separate lines in the Feynman graphs. It would be more relevant to say that the external momentum q is transmitted from the incoming to the outgoing photon through a coherent soft field fluctuation, see Fig. 4. An example is provided, for instance, by a fixed-size instanton. It is clear that this mechanism is conceptually related to the truncated tail of the condensate series. Indeed, as one proceeds to higher condensates, more lines become soft. Eventually we arrive at the situation when all lines are soft. Mathematically, the exponential/oscillating contribution is related to the factorial divergence of the

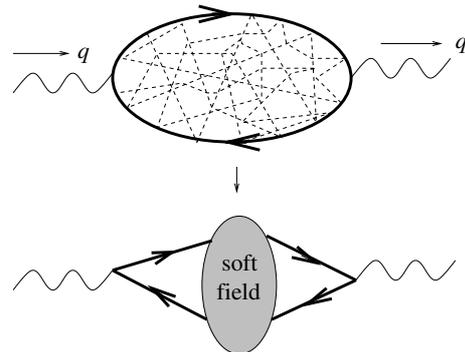


Figure 4. Transmitting a large external momentum through a soft field.

condensate series [3]. This is the usual story. The exponentially small terms in Euclidean convert into an oscillating function in Minkowski.

I will use the fixed-size instantons for the purpose of modeling this mechanism of duality violations. The observation that “soft” instantons generate an oscillating component ascends to Refs. [15,16]. By no means I imply that the instantons are the dominant soft fields in the QCD vacuum. True, there are models in which they are assumed to be dominant, the so-called instanton liquid models [17]. Their status in the range of questions I am interested in (the duality violations) has yet to be clarified. I will use instantons for the purpose of orientation, in hope that at least some features of the results obtained in this way will be more general than the model itself.

5 Model or theory?

The topic I address — the quark-hadron duality violations — has a unique status. By definition, one *cannot* build an exhaustive *theory* of the duality violations based on the first principles. Indeed, assuming there is a certain dynamical mechanism (which goes beyond perturbation theory and condensates) for which such a theory exists, one will immediately include the corresponding component in the theoretical calculation. The reference quantity, $\Pi(Q^2)_{\text{theor}}$, will be redefined accordingly. After the analytic continuation to Minkowski, this will lead, in turn, to a new theoretical spectral density to be used as a reference $\rho(s)_{\text{theor}}$ in the duality relation.

Thus, by the very nature of the problem, it is bound to be treated in models of various degrees of fundamentality and reliability. This is because the duality violation parametrizes our ignorance. Ideally, the models one should aim at must have a clear physical interpretation, and must be tested, in their key features, against experimental data. This will guarantee

a certain degree of confidence when these models are applied to the estimates of the duality violations in the processes and kinematical conditions where they had not been tested.

6 The physical picture behind the duality

Before delving into technical details I will describe the phenomenon from a slightly different perspective. The quark-hadron duality takes place in those processes where one can isolate two stages in the process under consideration, occurring at two distinct scales. A basic transition involving quarks (gluons) must typically occur at a short scale regulated by external parameters such as Q , m_Q , etc. For instance, in the e^+e^- annihilation the basic transition is the conversion of the virtual γ into $\bar{u}u$ or $\bar{d}d$. Then, at the second stage, the quarks (gluons) materialize in the form of hadrons, at a much larger scale. In the appropriate frame, the first time scale is of order $1/Q$ while the second of order Q/Λ^2 . By that time, the original quarks are far away from each other — a residual interaction cannot significantly alter the transition cross section which was “decided” at the first (quark-gluon) stage.

The duality violations are due to (i) rare atypical events, when the basic quark transition occurs at large rather than short distances; (ii) residual interactions occurring at large distances between the quarks produced at short distances. In the first case appropriate (Euclidean) correlation functions develop singularities at finite x^2 , while the second mechanism is correlated with the $x^2 \rightarrow \infty$ behavior.

In both cases the duality violating component follows the pattern I have discussed above — *exponential in Euclidean and oscillating in Minkowski*. Three distinct regimes were identified and considered in the literature so far:

(i) Finite-distance singularities

$$s^{-\kappa/2} \sin(\sqrt{s}); \quad (9)$$

(ii) Infinite-distance singularities ($N_c = \infty$)

$$s^{-\eta/2} \sin(s); \quad (10)$$

(iii) Infinite-distance singularities (N_c large but finite, $s \rightarrow \infty$)

$$\exp(-\alpha s) \sin(s), \quad \alpha = O\left(\frac{1}{N_c}\right) \ll 1. \quad (11)$$

These regimes are not mutually exclusive — in concrete processes one may expect the duality violating

component to be a combination of (i) and (ii), or (i) and (iii). From the theoretical standpoint it is quite difficult to consistently define the duality violating component of the type (iii). An operational definition I might suggest is as follows: Start from the limit $N_c = \infty$ and identify the component of the type (ii). Follow its evolution as N_c becomes large but finite.

7 An instanton-based model

The basic features of the formalism [4] to be used below to model exponential/oscillating terms (which present deviations from duality) are as follows:

(i) One considers quarks propagating in the instanton background field. The instanton size ρ is assumed to be fixed. Alternatively, one can say that an effective instanton measure has a δ -function peak in ρ .

(ii) For the light quarks, most instanton amplitudes are suppressed by powers of the light quark masses. This suppression is due to the fermion zero modes of the Dirac operator, occurring in the instanton-like backgrounds. We ignore these factors, as well as all other pre-exponential overall factors coming from the instanton measure. We will only trace dependences on large momenta relevant to the problems under consideration (Q in e^+e^- annihilation, the heavy quark mass m_Q in the case of the heavy quark decays, and so on).

(iii) We ignore all singularities of the correlation functions at $x = 0$. The corresponding contributions are associated with the power terms in the condensate expansions. The instantons-based models are presumably not precise enough to properly capture the condensates. We will isolate and calculate only those contributions that come from the finite distance singularities at $x^2 = -\rho^2$. In the momentum space they produce the exponential/oscillating terms sought for.

It is seen that our evaluation of deviations from duality is based on the most general aspects of the instanton formalism, and, in essence, does not depend on details.

The advantage of the model is its simplicity. The only formula we will need is

$$\int d^4x \frac{1}{(x^2 + \rho^2)^\nu} e^{iqx} = \frac{2\pi^2}{\Gamma(\nu)} \left(\frac{Q\rho}{2}\right)^{\nu-2} \frac{K_{2-\nu}(Q\rho)}{\rho^{2\nu-4}} \quad (12)$$

in the Euclidean domain, which implies that in the Minkowski domain, at large q^2 ,

$$\text{Im} \int d^4x \frac{1}{(x^2 + \rho^2)^\nu} e^{iqx} \propto s^{\frac{\nu}{2}-\frac{5}{4}} \sin(\sqrt{s}\rho - \delta), \quad s = q^2. \quad (13)$$

Here K is the McDonald function, and δ is a constant phase which is of no concern to us here.

To explain how it works it seems best to consider concrete examples. Let us start from e^+e^- annihilation. The polarization operator defined in Eq. (2) is given by the graph of Fig. 2. In Sec. 3 we evaluated this graph for free quarks and found that $\rho(s)$ tends to a constant at asymptotically large energies. Now, we evaluate the very same graph using the quark Green functions in the background instanton field, rather than the free quark Green functions. For massless quarks the Green functions G_{inst} are known exactly [18], but we will not need the full expression. We will only need to know that G_{inst} is a sum of terms of the following structure

$$G_{\text{inst}}(x, y) = \frac{1}{[(x-y)^2]^2} \times \frac{1}{[(x-z)^2 + \rho^2]^{\ell_1} [(y-z)^2 + \rho^2]^{\ell_2}} \tilde{G}, \quad (14)$$

where z is the instanton center, and

$$\ell_{1,2} = \frac{1}{2} \text{ or } \frac{3}{2}, \quad \ell_1 + \ell_2 = 2, \quad (15)$$

and the numerator \tilde{G} is a polynomial of x, y . As was explained above, the singularity of $G_{\text{inst}}(x, y)$ at $x - y = 0$ is irrelevant for the exponential/oscillating terms. Of importance are the singularities in the complex plane coming from the second factor in Eq. (14). The integrals which one has to take can (and must) be evaluated at a saddle point; a simple analysis [4] shows that at the saddle point the instanton center z is exactly in the middle between x and y . Then the relevant singularities of the quark correlation functions are at $(x - y)^2 = -4\rho^2$ (see Eq. (14)). At the singularity the first factor $1/(x - y)^4$ (which is singular at the origin) can be replaced by $1/(16\rho^4)$ and then safely omitted together with all other prefactors.

We can proceed now to the calculation of the exponential/oscillating component of the polarization operator $\Pi_{\mu\nu}$. The polarization operator is the product of two Green functions (14); therefore, at large Euclidean momenta

$$\begin{aligned} \Pi_{\mu\nu} &\propto \int d^4x e^{iq(x-y)} d^4z \\ &\times \frac{1}{[(x-z)^2 + \rho^2][(y-z)^2 + \rho^2]^3} \\ &\propto \int d^4x e^{iqx} \frac{1}{x^2 + \rho^2} \times \int d^4z e^{iqz} \frac{1}{(z^2 + \rho^2)^3} \\ &\propto K_1(Q\rho) K_{-1}(Q\rho) \propto \frac{1}{Q} \exp(-2Q\rho). \end{aligned} \quad (16)$$

Note that once the integration over the instanton center is carried out, the integral factorizes.

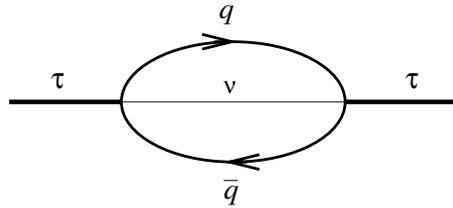


Figure 5. The transition operator \hat{T} determining the hadronic τ width, $\Gamma_{\text{hadr}}(\tau) = M_\tau^{-1} \text{Im} \langle \tau | \hat{T} | \tau \rangle$.

Equation (16) implies that the exponential component of $\Pi(q^2)$, defined in Eq. (3), is

$$\Delta\Pi \propto \frac{1}{Q^3} \exp(-2Q\rho), \quad (17)$$

which implies, in turn, that at large $E = \sqrt{s}$ the oscillating component of the spectral density is

$$\Delta\rho(s) \propto \frac{1}{E^3} \sin(2E\rho). \quad (18)$$

Equation (18) reproduces the high-energy asymptotics of the exact result [19] for the polarization operator in the one-instanton approximation.

Does the $s^{-3/2}$ fall off of the oscillating (duality violating) component make sense? I will confront theoretical expectations with experiment in Sec. 11. Here I just refer the reader to Fig. 10 (see the dashed curve), postponing a more detailed discussion till after both, the instanton-based and the resonance-based models, are considered.

The next example to be analyzed is the total hadronic τ width. This exercise is quite similar to previous exercises. The corresponding transition operator is depicted in Fig. 5 (its imaginary part at $p^2 = M_\tau^2$ is proportional to the width of the hadronic τ decay). The neutrino Green function clearly does not “feel” the background gluon field, so that the neutrino propagator is that of a free fermion. The same is valid for the τ lines. Therefore, we immediately conclude that

$$\Delta\Gamma(\tau \rightarrow \nu + \text{hadr.}) \propto \frac{1}{M_\tau} \sin(2M_\tau\rho), \quad (19)$$

cf. Eq. (16). The asymptotic (parton-model) prediction in this case is $\Gamma(\tau \rightarrow \nu + \text{hadr.}) \propto M_\tau^5$, so that the oscillating component in R_τ scales as

$$\Delta R_\tau \propto \frac{1}{M_\tau^6} \sin(2M_\tau\rho), \quad (20)$$

where

$$R_\tau \equiv \frac{\Gamma(\tau^- \rightarrow \nu_\tau + \text{hadrons})}{\Gamma(\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e)}. \quad (21)$$

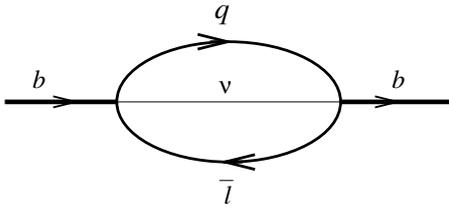


Figure 6. The transition operator \hat{T} determining the total semileptonic width of B mesons.

Note that the pre-exponential suppression factor in ΔR_τ is significantly stronger than in $\Delta R(e^+e^-)$, namely, M_τ^{-6} vs. E^{-3} . Of course, to make quantitative statements it is not sufficient to establish the scaling laws; one needs absolute normalizations. Here, we are basically in uncharted waters. At best, we have [4] some educated guesses, which, if true, imply that $\Delta R_\tau/R_\tau \lesssim 5\%$. This estimate is rather close to what one obtains for duality violations in τ in the resonance-based model, see Sec. 9.

In Sec. 2 I mentioned that the current interest to the problem of the quark-hadron duality was driven, to a large extent, by a significant progress in the experiments on the inclusive heavy flavor decays. Theoretically, they can be treated along the same lines as e^+e^- annihilation or τ decays. The distinctions are technical. Let us start from the total semi-leptonic width of the b flavored hadrons. At the quark level the process is described by the transition

$$b \rightarrow q\ell^- \nu,$$

where q is either u or c quark, and ℓ stands for the electron, muon or τ lepton. We will first neglect the masses of the final fermions; this is an excellent approximation for $q = u$ and $\ell = e, \mu$. The impact of the final quark (lepton) masses will be considered later.

The relevant transition operator is depicted in Fig. 6. The differences compared to the case of the τ decay are as follows:

- (i) There is one (rather than two) light-quark line carrying a large external momentum;
- (ii) Unlike the τ lines, the b quark lines do experience interactions with the background gluon field.

The analysis of the exponential/oscillating component in the heavy flavor decays combines standard elements of the heavy quark expansion (for a review see e.g. Ref. [20]), and the instanton calculus (a pedagogical review can be found in Ref. [21]). It is convenient to choose the rest frame of the heavy hadron at hand, and single out the large “mechanical” part in the x dependence of the heavy quark field,

$$Q(x) = e^{-im_Q t} \tilde{Q}(x).$$

Then the total width is proportional to the imaginary part of the transition operator,

$$\Gamma = \frac{1}{M_{H_Q}} \langle H_Q | \hat{T} | H_Q \rangle, \quad (22)$$

where

$$\hat{T} = i \int \bar{\tilde{Q}}(x) G(x, y) \tilde{Q}(y) D(x - y) e^{im_Q(x_0 - y_0)} d^4(x - y) d^4 z, \quad (23)$$

$G(x, y)$ is the light quark Green function, $D(x - y)$ describes the propagation of colorless objects (the lepton pair in the case at hand), while z is the instanton center. The subscript 0 marks the time component.

Both, the light quark Green function $G(x, y)$ and the heavy quark fields \tilde{Q} are to be considered in the background gluon field. Taken separately, they are not gauge invariant; the product in Eq. (23) is. In the leading order in the heavy quark expansion, the heavy quarks propagate only in time; therefore,

$$\tilde{Q}(x_0, \vec{x}) = T e^{i \int_0^{x_0} A_0(\tau, \vec{x}) d\tau} \tilde{Q}(0, \vec{x}) \equiv U(x) \tilde{Q}(0, \vec{x}). \quad (24)$$

It is convenient (although not necessary) to impose the condition that at large distances from the instanton center the quark propagation becomes free. This condition implies that the singular gauge is used. An explicit expression for $U(x)$ can be found in this gauge; it is rather cumbersome, and I will not quote it here, referring the reader to the original publication [4]. At the saddle point (which again corresponds to the instanton situated exactly in the middle between the points, x and y , i.e. $z = (1/2)(x + y)$) the product $U^{-1}(x) \dots U(y)$ in Eq. (23) reduces to unity, at least in the part which is singular at $(x - y)^2 = -4\rho^2$. Note, that the matrices converting the nonsingular-gauge Green function $G(x, y)$ to the singular gauge can be ignored too. This implies that one may continue to use Eq. (14).

Thus, the heavy quarks decouple from the instanton background in the calculation of the exponential/oscillating terms. This is the consequence of the fact that we exploit the heavy quark expansion and limit ourselves to the leading in $1/m_Q$ terms. In the subleading terms this decoupling does not necessarily take place. Generally speaking, the replacement of the z integral by the value of the integrand at the saddle point is not warranted in the next-to-leading orders. This effect will not be further pursued, however.

Summarizing, the heavy quark expansion makes the heavy quarks sterile with respect to the duality violating component. This is certainly not counterintuitive. The duality violating component in the heavy

quark inclusive decays originates from the light-quark propagators. Its calculation reduces to Eq. (14) and the routine outlined after this equation (plus the basic formula (12)). Starting from Eq. (23) we arrive at

$$\hat{T} \propto \frac{1}{m_Q^3} e^{-2m_Q \rho} \quad (25)$$

in the Euclidean domain, which results in

$$\Delta\Gamma_{\text{sl}} \propto \frac{1}{m_Q^3} \sin(2m_Q \rho). \quad (26)$$

Since the total width scales as $\Gamma \propto m_Q^5$, the oscillating component of the semileptonic branching ratio is obviously suppressed by m_Q^8 ,

$$\Delta\text{Br}(H_Q \rightarrow \ell\nu + \text{hadrons}) \propto \frac{1}{m_Q^8} \sin(2m_Q \rho). \quad (27)$$

It is easy to generalize this formula to include an arbitrary number of the light quarks in the final state. Each extra light quark adds two powers of m_Q in the numerator. Thus, in the total nonleptonic branching ratio, with three light quarks in the final states,

$$\Delta\text{Br}(H_Q \rightarrow \text{light hadrons}) \propto \frac{1}{m_Q^4} \sin(2m_Q \rho). \quad (28)$$

For the sake of completeness I will also mention the radiative decays of the type $b \rightarrow s + \gamma$. Assuming that these decays are induced by local operators²

$$\bar{b}\sigma_{\mu\nu}(1 + \gamma_5)sF_{\mu\nu},$$

where $F_{\mu\nu}$ is the photon field strength tensor, we immediately conclude that

$$\Delta\Gamma(b \rightarrow s + \gamma) \propto \frac{1}{m_Q^3} \sin(2m_Q \rho), \quad (29)$$

precisely in the same way as in Eq. (26). The parton expression for $\Gamma(b \rightarrow s + \gamma)$ scales as

$$\Gamma_0(b \rightarrow s + \gamma) \propto m_Q^3, \quad m_Q \rightarrow \infty. \quad (30)$$

As a result, our instanton-based model predicts that the duality violating component in $b \rightarrow s + \gamma$ is suppressed as

$$\Delta\Gamma(b \rightarrow s + \gamma)/\Gamma_0(b \rightarrow s + \gamma) \propto \frac{1}{m_Q^6} \sin(2m_Q \rho). \quad (31)$$

So far it was assumed that the quarks and leptons produced were massless. What changes if we decide to take into account finite (nonvanishing) masses of the quarks and leptons?

²In actuality this is true only for a part of the amplitude.

$e^+e^- \rightarrow X$	$\tau \rightarrow \nu X$	$H_Q \rightarrow \ell\nu X$
$\frac{1}{E^3}$	$\frac{1}{M_\tau^6}$	$\frac{1}{M_{H_Q}^8}$
$H_Q \rightarrow X$	$H_Q \rightarrow X\gamma$	$H_Q \rightarrow H_{Q'}X$
$\frac{1}{M_{H_Q}^4}$	$\frac{1}{M_{H_Q}^6}$	$\frac{1}{(\Delta M_{H_Q})^6}$

Table 1. The index of the power fall-off of the oscillating (duality violating) component in various inclusive processes, normalized to the corresponding asymptotic (parton model) formulae, in the instanton-based model. The capital X denotes the light-quark hadronic states.

The exact dependence on the masses of the final quarks or leptons is rather sophisticated. Although it is calculable in principle, the corresponding calculations are much harder to perform than the simple estimates presented above. Moreover, this is hardly necessary. Given a crude nature of the model, which is intended for orientation rather than for precision estimates, it seems reasonable to treat u, d and s quarks as massless while c as heavy, and the c lines as free propagation. Then the only impact of the c quark mass is kinematical, it results in the replacement of the total energy m_Q by a relevant energy release in the light quarks in the process at hand.

These are just a few of important applications which are under discussion in the current literature. The results are collected in Table 1. Our instanton-based model of the duality violation is user-friendly — it is very easy to evaluate the exponential/oscillating component in other inclusive processes not included in Table 1, would such a necessity arise.

8 Global vs. local duality

Usually by local duality people mean point-by-point comparison of $\rho(s)_{\text{theor}}$ and $\rho(s)_{\text{exp}}$, while global duality compares the spectral densities $\overline{\rho(s)}$ averaged over some *ad hoc* interval of s , with an *ad hoc* weight function $w(s)$,

$$\int_{s_1}^{s_2} ds w(s) \rho(s)_{\text{theor}} \approx \int_{s_1}^{s_2} ds w(s) \rho(s)_{\text{exp}}.$$

Here I would like to comment on a very common misconception which travels from one paper to another. Many authors believe that global duality defined in this way has a more solid status than local duality. Some authors go so far as to say that while global duality is certainly valid at high energies, this is not

necessarily the case for local duality. This became a routine statement in the literature. Well, routine does not mean correct.

In fact, both procedures have exactly the same theoretical status. The point-by-point comparison, as well as the comparison of $\overline{\rho(s)}$'s (with an *ad hoc* weight function), must be considered as distinct versions of local duality. The distinction between the “local” quantities, such as $R(e^+e^-)$ at a certain value of s and the integrals of the type involved, say, in R_τ is quantitative rather than qualitative. Comparison of Eqs. (18) and (20) makes this assertion absolutely transparent: in both quantities there is a duality violating component, the only distinction is a concrete index of the power fall-off (3 vs. 6).

The genuine global duality applies only to special integrals which can be *directly expressed through the Euclidean quantities*. For instance, if the integration interval extends from zero to infinity, and the weight function is exponential, the integral

$$\int_0^\infty ds \exp\{-s/M^2\} \rho(s),$$

reduces [14] to the Borel transform of the polarization operator $\Pi(Q^2)$ in the Euclidean domain (i.e. at positive Q^2). For such quantities, duality *cannot* be violated, by definition.

There is one more aspect of averaging (smearing) which is not fully understood in the literature and requires comment. Let us consider again R_τ . It can be expressed in terms of spectral densities ρ_V and ρ_A in the vector and axial-vector channels, respectively,

$$R_\tau = \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + 2\frac{s}{M_\tau^2}\right) [\rho_V(s) + \rho_A(s)]. \quad (32)$$

(More exactly, the integration over s runs from M_π^2 , but in the chiral limit we stick to, the pion mass vanishes.) The spectral densities ρ_V and ρ_A are normalized in such a way that their asymptotic (free-quark) values are

$$\rho_V(s), \rho_A(s) \rightarrow N_c \quad \text{at} \quad s \rightarrow \infty.$$

Correspondingly, the asymptotic value of R_τ is $R_\tau^0 = N_c$.

In the instanton-based model, the duality violating component in $\rho_{V,A}$ scales as E^{-3} . It is tempting to say then, that the duality violating component in R_τ can be obtained by integrating the duality violating components of $\rho_{V,A}$ with the weight function specified in Eq. (32).

This would lead us nowhere, however. First of all, the integral in Eq. (32) runs all the way down to zero, while Eq. (18) is valid at asymptotically large E . Even if the lower limit of integration were chosen to be high enough, this would not help to find ΔR_τ from Eq. (32). Indeed, $\Delta\rho_{V,A}$ is an oscillating function of s , any smearing inevitably entails cancellations, and the result would depend on subtle details of $\rho(s)$, which are certainly beyond theoretical control. The cancellations are seen, in particular, from the fact that asymptotically $\Delta R_\tau(M_\tau)$ falls off faster than $\Delta\rho_{V,A}(s)$, namely, M_τ^{-6} versus E^{-3} . There is no way one could predict the change of the index from 6 to 3 based solely on the integral (32).

At the same time, our instanton-based model does allow one to predict the index in R_τ . To this end one must analyze the appropriate transition amplitude in the Euclidean domain performing the analytic continuation to the Minkowski domain at the very end. The model captures enough intricate features of QCD to “know” the result of the smearing as a whole. Note that $\Delta R_\tau(M_\tau)$ depends on the highest scale in the problem at hand, M_τ , rather than on any intermediate or low scales one encounters in process of integration in Eq. (32). This is a general feature which will be valid in any process and will persist in any model based on evaluating singularities off the origin.

9 A resonance-based model

Now we will acquaint ourselves with another approach based on the resonance saturation of the colorless n -point functions. That this is a distinct dynamical source of duality violations follows from the discussion in Secs. 4 and 6. Shortly we will confirm this by observing that the functional form of oscillations comes out different compared to that in the instanton-based model. Theoretical analysis is most transparent in the limit $N_c = \infty$. Later I will allow N_c to become finite albeit large (Sec. 10).

Let us first summarize what we expect to get for the polarization operator defined in Eq. (2) in the limit of large N_c . In multicolor QCD, $N_c \rightarrow \infty$, the resonance widths vanish. The spectrum of excitations in the given channel is expected to be (asymptotically) equidistant. A string-like picture of the color confinement naturally leads to (approximately) linear Regge trajectories. For each primary trajectory there are infinitely many daughter ones. The daughter trajectories are parallel to the primary trajectory and are shifted by integers (for a review see e.g. Ref. [22]). As a result, the excitation spectrum in the given channel takes the form

$$M_n^2 = M_0^2 + \sigma^2 n, \quad \sigma^2 \equiv 2/\alpha', \quad (33)$$

where α' is the slope of the trajectories. Note that the neighboring resonance states in the given channel are separated by the interval $2/\alpha'$ rather than $1/\alpha'$. This is due to the alternating signatures of the daughter trajectories.

All these properties can be extracted from the Veneziano amplitude found in 1968 (see Sec. 7.4 in Collins [22]), which gave rise to the modern string theory.

Equation (33) implies, in turn, that $\Pi(q^2)$ can be presented as an infinite sum

$$\begin{aligned}\Pi(q^2) &= -\frac{N_c\sigma^2}{12\pi^2} \sum_{n=0}^{\infty} \frac{1}{q^2 - M_n^2} \\ &= -\frac{N_c\sigma^2}{12\pi^2} \sum_{n=0}^{\infty} \frac{1}{q^2 - \sigma^2 n - M_0^2}.\end{aligned}\quad (34)$$

In the problem at hand there are two large quantities: the number of colors and the energy. If one fixes s and lets N_c grow, one eventually arrives at the comb of infinitely narrow δ functions, as in Fig. 7. On the other hand, if N_c is fixed (no matter how large it is), with increasing s one eventually finds oneself in the energy region where the resonance widths cannot be neglected; in fact, the resonances start overlapping, and $\rho(s)$ gets smoothed. The approximation of the infinitely narrow resonances badly fails here, and must be amended. These two limits ($N_c \rightarrow \infty$ with s fixed, and N_c fixed with $s \rightarrow \infty$) are not interchangeable. Later I will include the nonvanishing widths (Sec. 10). For the time being I put $N_c = \infty$, so that all resonance widths vanish.

The equidistant spectrum only holds if both, the primary and all daughter trajectories, are exactly linear and parallel. This is not fully realistic. Even putting $N_c = \infty$ does not help. The low-energy parts of the Regge trajectories in QCD are not exactly linear, in particular, due to the spontaneous breaking of the chiral symmetry and the emergence of the massless pions.

This shortcoming of the model affects the condensate expansion at low orders but has no impact on deviations from duality at high energies. Since the phenomenon under discussion is related to the factorial divergence of the high-order terms, letting the low-lying excitations “breathe” does not change the factorial behavior [23], which is in one-to-one correspondence with the spectral formula at large n , where the spacings $M_{n+1}^2 - M_n^2$ must be constant. Therefore, it is okay to use the linear pattern (34). For the very same reason the residues of all resonances in Eq. (34) are taken to be equal. Fluctuations of the residues would show up in the condensate expansion; they do

not affect the estimate of the duality violations. (For an additional remark on the residues see Sec. 11.)

The infinite sum in Eq. (34) reduces to a well-known Euler’s ψ function, the logarithmic derivative of Γ ,

$$\Pi(Q^2) = -\frac{N_c}{12\pi^2} [\psi(z) + \text{Const}], \quad z \equiv \frac{Q^2 + M_0^2}{\sigma^2}, \quad (35)$$

(see e.g. Ref. [24]). The irrelevant (subtraction) constant on the right-hand side is infinite, strictly speaking. The occurrence of the Γ function reminds us of Veneziano’s amplitude.

At positive values of Q^2 an asymptotic representation exists for the ψ function,

$$\psi(z) = \ln z - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} z^{-2k}, \quad (36)$$

where B_{2k} stand for the Bernoulli numbers,

$$B_{2k} = (-1)^{k-1} \frac{2(2k)!}{(2\pi)^{2k}} \zeta(2k); \quad (37)$$

here ζ is the Riemann function. (In some textbooks $(-1)^{k+1} B_{2k}$ is called the k -th Bernoulli number and is denoted by B_k .) Equation (36) defines the asymptotic expansion of the polarization operator,

$$\Pi(Q^2) \xrightarrow{Q^2 \rightarrow \infty} -\frac{N_c}{12\pi^2} \ln Q^2 + \sum_{k=1}^{\infty} \frac{C_k}{(Q^2)^k}, \quad (38)$$

where the coefficients C_k can be expressed through B_{2k} (see Eq. (36)) in a relatively straightforward manner. The leading logarithmic term exactly coincides with the free quark loop of Fig. 2 presented in Eq. (5). This explains our choice of the resonance residues. The next-to-leading term is $1/Q^2$, followed by higher power corrections. Equations (36) and (37) highlight the factorial divergence of the condensate series which I have already mentioned several times.

One might want to eliminate the $1/Q^2$ term from $\Pi(Q^2)$ to make the $1/Q^2$ expansion realistic — it is known [14] that, in QCD with massless quarks, the terms of the first order in $1/Q^2$ do not appear in $\Pi(Q^2)$. The power series starts from $\langle G^2 \rangle / Q^4$ where $\langle G^2 \rangle$ is the gluon condensate. One could eliminate $1/Q^2$, say, by fine-tuning the parameter M_0^2 . If $M_0^2 = \sigma^2/2 = 1/\alpha'$, the $1/Q^2$ term cancels. Although this might seem desirable, in fact this is hardly worth the bother because the model with exactly linear trajectories is too rigid to be realistic anyway. The $1/Q^4$ correction will come out way too large. To accommodate the gluon and mixed condensates properly one would need a more flexible model, with more than one adjustable parameter. Therefore, I will feel free to

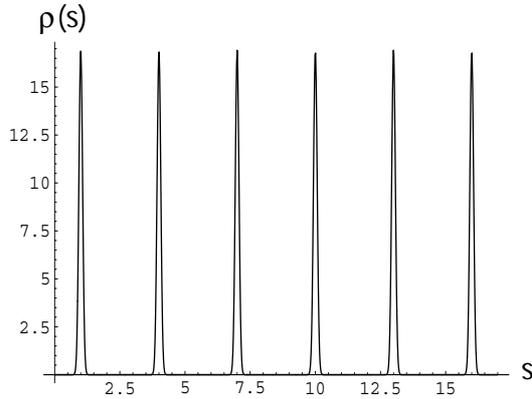


Figure 7. The spectral density in the resonance model. For clarity I gave a tiny width to the δ functions.

simplify the model further by putting $M_0^2 = 0$ and discarding the term with $n = 0$ in Eq. (34).

The spectral density corresponding to the infinite sum of the equidistant resonances is shown in Fig. 7,

$$\rho_V(s) = \rho_A(s) = N_c \cdot \sum_{n=1}^{\infty} \delta\left(\frac{s}{\sigma^2} - n\right); \quad \sigma^2 = \frac{2}{\alpha'}.$$
 (39)

Let us truncate the power expansion (38) at some finite order, and examine the theoretical prediction for $\rho(s)$ obtained from the Euclidean side. It does not matter in which order we truncate. Any power term $(1/Q^2)^n$ is invisible in $\rho(s)$ at positive s : analytically continuing to positive q^2 and taking the imaginary part one ends up with the δ function and its derivatives. The only imaginary part at positive s comes from the analytic continuation of $\ln Q^2$. Thus, in the model at hand $\rho(s)_{\text{theor}} = 1$. Comparing this with the comb of the δ functions in Fig. 7 we conclude that the point-by-point duality is maximally violated: between the resonances the spectral density is grossly overestimated (the “experimental” curve runs below the “theoretical” expectation) while at the peaks it is grossly underestimated (the “experimental” curve runs above the “theoretical” expectation). What is most crucial, the deviations from duality do not die off with energy. The power index vanishes!

All this is pretty trivial. A somewhat less trivial question worth examining is the impact of the *ad hoc* smearing. For instance, let us average the spectral density in Fig. 7 with the weight function appropriate to R_τ ,

$$R_\tau = \frac{I_0(M_\tau^2)}{M_\tau^2} - 3 \frac{I_2(M_\tau^2)}{M_\tau^6} + 2 \frac{I_3(M_\tau^2)}{M_\tau^8}, \quad (40)$$

where the moments I_n are defined as

$$I_n(M) = \int_0^{M^2} ds s^n [\rho_V(s) + \rho_A(s)]. \quad (41)$$

To estimate the oscillating contribution to R_τ which constitutes duality violation that cannot be seen in a truncated OPE we treat M_τ as a free (large) parameter. It will be seen momentarily that for $N_c = \infty$ and M_τ large, yet finite, the duality violation in R_τ scales as $1/M_\tau^6$. In other words, the vanishing power index in ρ translates³ in $\eta = 6$ in R_τ . Certainly, this distinction is quantitative rather than qualitative, but, sure enough, it is quite important from the practical side.

The sum over resonances in R_τ is easily calculated analytically: for the spectral density of Eq. (39) it is

$$\begin{aligned} R_\tau &= R_\tau^{\text{OPE}} + \Delta R_\tau^{\text{osc}}, \\ \frac{R_\tau^{\text{OPE}}}{N_c} &= 1 - \frac{\sigma^2}{M_\tau^2} + \frac{1}{30} \left(\frac{\sigma^2}{M_\tau^2}\right)^4, \\ \frac{\Delta R_\tau^{\text{osc}}}{N_c} &= -x(1-x)(1-2x) \left(\frac{\sigma^2}{M_\tau^2}\right)^3 \\ &+ \left[x^2(1-x)^2 - \frac{1}{30}\right] \left(\frac{\sigma^2}{M_\tau^2}\right)^4, \end{aligned} \quad (42)$$

where

$$x = \text{fractional part of } \left(\frac{M_\tau^2}{\sigma^2}\right), \quad x \in [0, 1].$$

I presented the result as a sum of two functions of M_τ^2 . The first one, R_τ^{OPE} , is a smooth function expandable in $1/M_\tau^2$. The second one, $\Delta R_\tau^{\text{osc}}$, oscillates with the period σ^2 ; its average vanishes, see the plot of $\Delta R_\tau^{\text{osc}}/R_\tau^0$ in Fig. 8. Although $\Delta R_\tau^{\text{osc}}$ is not a pure sine — it contains higher harmonics — the coefficients of the higher harmonics are numerically suppressed. With the accuracy of a few percent one can write

$$\frac{\Delta R_\tau^{\text{osc}}}{R_\tau} = -\frac{1}{3\sqrt{12}} \left(\frac{\sigma^2}{M_\tau^2}\right)^3 \sin\left(2\pi \frac{M_\tau^2}{\sigma^2}\right). \quad (43)$$

I pause here to make a comment. The contribution of any particular resonance of mass M_k to R_τ , according to Eq. (32), is given by a simple polynomial in $1/M_\tau^2$ (times the step function $\theta(M_\tau^2 - M_k^2)$). Variations of parameters of a given resonance (or resonances) change only the regular terms of the $1/M_\tau^2$ expansion, but have no impact on the oscillatory component. From Eq. (32) it is clear that such variations

³I remind that the power index η was defined in Sec. 4; in the case at hand $\Delta R_\tau/R_\tau \sim M_\tau^{-\eta} \sin M_\tau^2/\sigma^2$.

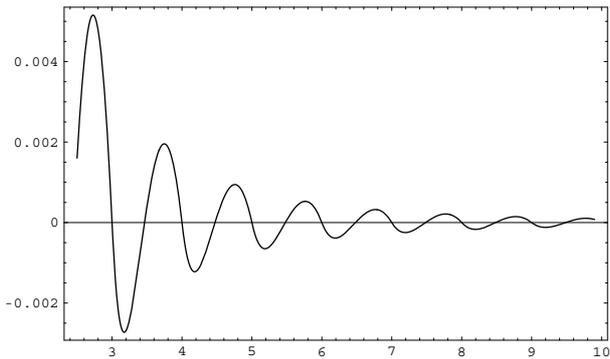


Figure 8. Oscillations in R_τ . The plot of $\Delta R_\tau^{\text{osc}}/R_\tau^0$ is presented as a function of M_τ^2/σ^2 .

change only coefficients of the $1/M_\tau^2$, $1/M_\tau^6$ and $1/M_\tau^8$ terms. OPE for R_τ must exactly reproduce these three expansion coefficients, and so it does.

In fact, it is not difficult to demonstrate that R_τ^{OPE} coincides with the OPE prediction in the model at hand. The power corrections can be presented as follows:

$$R_\tau^{\text{OPE}} = N_c + \frac{\tilde{I}_0}{M_\tau^2} - 3\frac{\tilde{I}_2}{M_\tau^6} + 2\frac{\tilde{I}_3}{M_\tau^8}, \quad (44)$$

where the “condensates” \tilde{I}_n are

$$\tilde{I}_n = \int_0^\infty ds s^n [\rho_V(s) + \rho_A(s) - 2N_c]. \quad (45)$$

These integral representations for the “condensates” \tilde{I}_n follow from Eqs. (32), (41) if one assumes that the spectral densities approach their asymptotic limits faster than any power of $1/s$. In the model at hand, with the comb-like spectral density, the integral representation (45) requires regularization. As a regularization one can introduce the weight factor $\exp(-\epsilon s)$, taking the limit $\epsilon \rightarrow 0$ at the end. With this regularization, R_τ^{OPE} from Eq. (42) is reproduced.

The same result for the coefficients of the power terms in R_τ^{OPE} could be obtained directly from the expansion of $\Pi(Q^2)$ in Eq. (35). We agreed to put $M_0^2 = 0$ for simplicity. Then the expansion coefficients of $\Pi(Q^2)$ are those of the ψ function in Eq. (36). To find the power terms in R_τ one may analytically continue the power terms in Eq. (36) to Minkowski, take the imaginary part and convolute with the weight function presented in Eq. (32). It is easy to see that the only relevant terms in Eq. (36) are $1/z$ and $1/z^4$ giving rise to $\delta(s)$ and $\delta'''(s)$ in the imaginary part. The terms of the higher order in $1/z$ drop out because the weight function is a polynomial of the third order; it contains no s^4 or higher terms. The term $1/z^2$ in Eq. (36) (it would generate $\delta'(s)$) drops out because the weight function does not contain terms linear in s . Then, the

expansion for R_τ^{OPE} is immediately recovered provided that one substitutes the appropriate value for the appropriate Bernoulli number, $B_2 = -1/30$.

A few words on the numerical aspect. Our consideration is admittedly illustrative. One should not take too literally the numbers which ensue for many reasons: in particular, M_τ^2 is not much larger than the spacing between the resonances, $N_c = 3$ is probably not large enough to warrant the zero width approximation, and so on. I would not put too much confidence on particular numbers. I would settle on the statement that the above estimate of the oscillation component is valid, say, up to a factor of two or so. Taking our formula for $\Delta R_\tau^{\text{osc}}$ at its face value and using the actual value of the τ mass ($M_\tau^2/\sigma^2 \sim 1.5$) we obtain that $\Delta R_\tau^{\text{osc}}/R_\tau \sim 3\%$. It is rather rewarding to see that this estimate is in the same ball park as that obtained in the instanton-based model. It seems safe to conclude that duality violations in the total hadronic τ width are expected at a level of a few percent. We do not know whether two mechanisms add constructively or destructively. Given this additional uncertainty, it will be no exaggeration to assert that the overall theoretical uncertainty

$$\Delta R_\tau/R_\tau = 3\%.$$

The 3% uncertainty in the hadronic τ width translates into $\sim 20\%$ uncertainty in $\alpha_s(M_\tau)$, which entails the uncertainty of about 6% in $\alpha_s(M_Z)$.

In summary:

- The resonance structure of the hadronic spectrum associated with the confining properties of QCD leads to a distinct exponential/oscillating component invisible in the truncated OPE.
- This component is related to singularities of the appropriate n -point functions at infinite separations (in the coordinate space).
- The corresponding duality violations are maximal at $N_c = \infty$, when the resonance widths vanish. The power index which was introduced in Sec. 4 is calculable in many instances. Generally speaking, the power indices obtained in the instanton-based and resonance-based models do not coincide with each other. R_τ seems to be an exceptional case. We do not understand the reasons explaining the coincidence of the power indices in R_τ ; probably, it is accidental.
- Inclusion of the nonvanishing resonance widths will replace the power suppression in the pre-exponent by a weak exponential suppression, see Sec. 10. The change of the regimes will probably have little numerical impact in the τ decays since M_τ^2 is not large enough for the exponential regime to develop in earnest. The

factor $(\sigma^2/M_\tau^2)^3$ in Eq. (43) will be replaced by

$$\exp\left(-\frac{2\pi BM_\tau^2}{N_c\sigma^2}\right).$$

Both factors are rather close numerically.

10 The impact of the resonance widths

As was already mentioned, no matter how large N_c is, at sufficiently high energies one cannot neglect the resonance widths. One should also remember that in the real world the number of colors is three — not very large by any count. One can expect that the nonvanishing resonance widths, once the resonances start to overlap, provide an additional suppression of the duality violating component.

Below I will show that, inside the domain of overlapping, the power regime (10) is replaced by an exponential one, see Eq. (11), even if the power index vanishes (as is the case for the spectral density (4) presented in Fig. 7). The basic idea is that the nonvanishing resonance widths shift the poles away from the physical cut, to unphysical sheets, which automatically results in a smoother imaginary part on the physical cut, much closer to that obtained from OPE.

I return to the equidistant resonances considered in Sec. 9 in the leading order in $1/N_c$, with the intention to include the next-to-leading $1/N_c$ effects. For any given excitation n , the widths scale as $\Gamma_n/M_n \sim N_c^{-1}$. Below we will analyze an infinite sequence of resonances for which we will need to know the dependence of this ratio on the excitation number n . For the time being, however, let us focus on an “isolated” resonance.

In the leading order its contribution to the polarization operator $\Pi(q^2)$ is

$$\frac{g_n^2}{q^2 - M_n^2}; \quad (46)$$

here g_n^2 is the residue. If $\Gamma_n \neq 0$, in the vicinity of the pole one can use the Breit-Wigner formula which replaces Eq. (46) by

$$\frac{g_n^2}{q^2 - M_n^2 + i\Gamma_n M_n}. \quad (47)$$

Strictly speaking, g_n^2 and M_n^2 in Eqs. (46) and (47) are different: in the former the residue and the mass must be taken in the leading order in $1/N_c$ while in the latter they include $O(1/N_c)$ corrections. This effect is less important than that due to Γ_n 's. In many instances (but not always, see below) the $1/N_c$ shifts of g_n^2 and M_n^2 can be neglected.

Now comes a crucial assertion. At large n the resonance width must scale as $\Gamma_n \sim M_n/N_c$, i.e. the n dependence of Γ_n is the same as that of M_n , the square root of n . This behavior was predicted long ago [25] on the basis of a simple qualitative picture; much later it was confirmed [5] by numerical studies in the two-dimensional 't Hooft model. This formula can be explained as follows. For highly excited states a quasiclassical treatment applies. When a meson is created by a local source, it can be considered, quasiclassically, as a pair of (almost free) ultrarelativistic quarks; each of them with energy $M_n/2$. These quarks are produced at the origin, and then fly back-to-back, creating behind them a flux tube of the chromoelectric field. The length of the tube $L \sim M_n/\Lambda^2$ where Λ^2 represents the string tension. The meson decay width is determined, to order $1/N_c$, by the probability of producing an extra quark-antiquark pair. Since the pair creation can happen anywhere inside the flux tube, one expects that⁴ $\Gamma_n \sim L\Lambda^2/N_c$. Taking into account that $L \sim M_n/\Lambda^2$ one arrives at

$$\Gamma_n = \frac{B}{N_c} M_n, \quad (48)$$

where B is a dimensionless coefficient of order one. Naively extrapolating Eq. (48) to $n = 1$ and $N_c = 3$ (well beyond the limits of its applicability) and normalizing by the ρ meson for which $\Gamma/M \sim 0.2$, one can make an educated guess,

$$B \sim 0.5. \quad (49)$$

The Breit-Wigner resonance formula now takes the form (in the Euclidean notation)

$$-g_n^2 \left[Q^2 + M_n^2 - i\frac{BM_n^2}{N_c} \right]^{-1}. \quad (50)$$

One must exercise certain care in using the Breit-Wigner expression for the resonance contribution in the polarization operator far away from the pole. It must be adjusted in such a way that the analytic properties of $\Pi(q^2)$ are not spoiled. Namely, $\Pi(q^2)$ must remain analytic everywhere in the complex q^2 plane, with a cut along the positive real semi-axis. No singularities are allowed on the physical sheet, all poles must be shifted to unphysical sheets. This is relatively easy to achieve. To this end let us replace Eq. (50) by

$$-\tilde{g}_n^2 \left[Q^2 \left(1 - \frac{B}{\pi N_c} \ln \frac{Q^2}{\sigma^2} \right) + \tilde{M}_n^2 \right]^{-1}. \quad (51)$$

On the upper side of the physical cut near the pole (i.e. at Q^2 in the vicinity of $-M_n^2 - i\epsilon$) the imaginary

⁴Let me note in passing that the $1/N_c^2$ corrections due to creation of two quark pairs are of order L^2/N_c^2 within this picture. Since $L \sim M_n \sim \sqrt{n}$, the expansion parameter is \sqrt{n}/N_c .

parts of both expressions (50) and (51) coincide, provided that \tilde{g}_n^2 and \tilde{M}_n^2 are appropriately adjusted (\tilde{g}_n^2 and \tilde{M}_n^2 differ from g_n^2 and M_n^2 by a $1/N_c$ correction; in what follows we will omit the tilde, making the adjustment at the very end). Thus, both expressions are equally legitimate for the Breit-Wigner description of the resonances. Moreover, Eq. (51), in turn, can be replaced by

$$-g_n^2 \left[Q^2 \left(\frac{Q^2}{\sigma^2} \right)^{-\frac{B}{\pi N_c}} + M_n^2 \right]^{-1}. \quad (52)$$

The distinction between Eqs. (51) and (52) is of the order $O(1/N_c^2)$. We do not track such terms anyway. This latter formula has the required analytic properties — it is nonsingular everywhere in the complex Q^2 plane, except the cut at negative real Q^2 . On the physical sheet of Q^2 , the variable

$$z \equiv \left(\frac{Q^2}{\sigma^2} \right)^{1 - \frac{B}{\pi N_c}} \quad (53)$$

never becomes real and negative, so that the pole singularities in Eq. (52) are indeed shifted to unphysical sheets.

Summing over the infinite chain of the equidistant resonances, as in Eq. (34), we arrive at

$$\begin{aligned} \Pi(Q^2) &= \frac{N_c \sigma^2}{12\pi^2} \frac{1}{1 - B/(\pi N_c)} \\ &\times \sum_{n=1}^{\infty} \left[Q^2 \left(\frac{Q^2}{\sigma^2} \right)^{-\frac{B}{\pi N_c}} + M_n^2 \right]^{-1} \\ &= -\frac{N_c}{12\pi^2} \frac{1}{1 - B/(\pi N_c)} \left[\psi(z) + \frac{1}{z} \right], \quad (54) \end{aligned}$$

where z is defined in Eq. (53), and $1 - B/(\pi N_c)$ in the denominator reflects the adjustment of the residues discussed above (this factor can be established by demanding the correct asymptotic behavior at $Q^2 \rightarrow \infty$, cf. Eq. (5)).

How does this compare with the zero-width approximation of Sec. 9? Formally both results for the polarization operator look similar; the only difference⁵ is in the definition of the variable z . In the deep Euclidean domain the power expansion of $\Pi(Q^2)$ ensues from the asymptotic representation (36). The k -th term of the expansion, which in the leading order in $1/N_c$ used to

⁵The $1/z$ term which was absent in Sec. 9 is due to the fact that I put $M_0^2 = 0$ and start the summation from $n = 1$ rather than from $n = 0$. These simplifications resulting in the occurrence of the $1/z$ term, are irrelevant in the studies of the duality violating component at high energies.

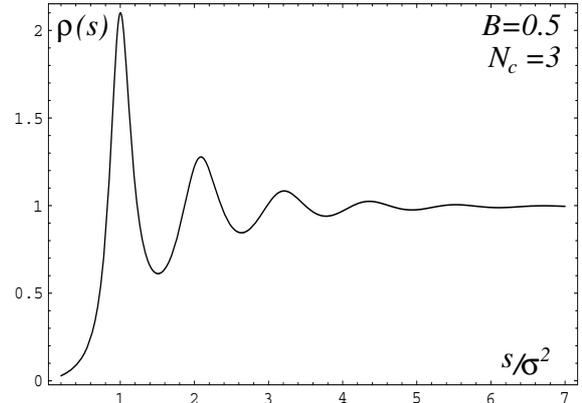


Figure 9. The spectral density for the equidistant resonances with finite widths specified in Eq. (48).

be

$$\frac{B_{2k}}{2k} \left(\frac{\sigma^2}{Q^2} \right)^{2k} \quad (\text{zero-width approx.}),$$

now becomes

$$\frac{B_{2k}}{2k} \left(\frac{\sigma^2}{Q^2} \right)^{2k} \left(1 + \frac{2kB}{\pi N_c} \ln \frac{Q^2}{\sigma^2} + \frac{B}{\pi N_c} + O\left(\frac{1}{N_c^2}\right) \right). \quad (55)$$

The impact of the next-to-leading $1/N_c$ terms on the power expansion is quite insignificant. In particular, the $(B/N_c) \ln Q^2$ correction mimics the logarithmic anomalous dimension typical of OPE in QCD.

At the same time, the $1/N_c$ effects radically affect $\text{Im}\Pi$ on the physical cut (the spectral density). Instead of the comb of the δ functions of Fig. 7, we now get at high energies a smooth function, with mild oscillations. The plot of the spectral density $\rho(s) = (12\pi/N_c) \text{Im}\Pi$ (with Π determined by Eqs. (54) and (53)) is presented in Fig. 9.

At asymptotically large s the spectral density tends to unity, plus power corrections corresponding to OPE plus an oscillating component which cannot be obtained in the power expansion. One can readily isolate it by exploiting the well-known reflection property of Euler's function,

$$\psi(z) + \frac{1}{z} = \psi(-z) - \pi \cot \pi z. \quad (56)$$

On the physical cut (in units of σ^2)

$$z = -s^{1 - \frac{B}{\pi N_c}} \left(1 + i \frac{B}{N_c} \right) + O\left(\frac{1}{N_c^2}\right), \quad (57)$$

and the imaginary part of the left-hand side of Eq. (56) is essentially given by that of $-\pi \cot \pi z$. The imaginary part of $\psi(-z)$ gives rise to non-oscillating

power corrections s^{-2k} in the spectral density corresponding to OPE. Alternatively, these OPE corrections can be obtained by a direct analytic continuation to Minkowski of the power terms (55).

As a result, the finite-width resonance model leads us to

$$\rho(s) \rightarrow 1 + \text{power corr.} + 2 \exp\left(-\frac{2\pi s B}{\sigma^2 N_c}\right) \cos\left(\frac{2\pi s}{\sigma^2}\right), \quad (58)$$

where it is assumed that

$$\frac{2\pi s B}{\sigma^2 N_c} \gg 1 \quad \text{but} \quad \frac{\ln s}{N_c} \ll 1. \quad (59)$$

If s is fixed while N_c is set to ∞ we recover unsuppressed oscillations, as in Sec. 7. In the opposite limit N_c fixed and s large we observe an exponential suppression, with a weak exponent proportional to $1/N_c$.

A key question one can ask in connection with the above analysis is as follows. For the given value of N_c , what is the boundary energy marking the onset of the exponential suppression? The best I can say at the moment is that this boundary energy scales as $s_0 \propto N_c$. A reliable determination of the proportionality coefficient is a task for the future. A naive estimate following from Eq. (59),

$$s_0 \sim \frac{\sigma^2 N_c}{2\pi B} \sim 2 \text{ GeV}^2 \quad \text{at} \quad N_c = 3,$$

if correct, would mean that the resonances essentially overlap (and, thus, smear the spectral density), starting from the first or second excitation.

11 How does all this match experiment?

In spite of the current extremely high demand on theoretical estimates of duality violations, there is surprisingly little effort to elucidate the issue by direct experimental studies. A high-precision measurement of the ratio R in e^+e^- in a broad range of energies, from threshold up to, say, $s = 10 \text{ GeV}^2$ in a dedicated experiment with the proper absolute normalization would give an enormous boost to this issue. Alas, such measurements have never been undertaken... The most accurate data on the spectral densities were obtained in τ decays [26], where they (naturally) extend only up to $s = 3 \text{ GeV}^2$. This energy range is way too narrow to be helpful in perfecting the models which are currently in use or in the design of new models. Still, confronting the data with current theoretic ideas might give a feeling of whether or not we are moving in the right direction. The comparison is presented in

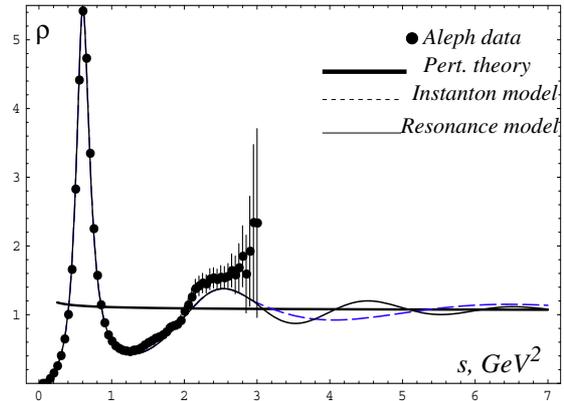


Figure 10. Experimental data (ALEPH) on the spectral density in the vector channel and the duality violation models.

Fig. 10. I will first explain what is depicted, and then offer several comments.

The ALEPH experimental data [26] I show correspond to the spectral density in the vector isovector channel. There exist some data above 3 GeV^2 from e^+e^- annihilation, but the error bars are so large that plotting these points would just obscure the picture. Directly measurable in the hadronic τ decays is the sum of the vector and axial spectral densities. To obtain the spectral densities separately one has to sort out all decays by assigning specific quantum numbers to each given final hadronic state. In the majority of cases such an assignment is unambiguous. For instance, two pions (whose contribution is the largest) can be produced only by the vector current. Some processes, however, can occur in both channels, for instance, the $KK\pi$ production. Using certain theoretical arguments it was decided [26] that around 3/4 of all $KK\pi$ yield must be ascribed to the vector channel. Other theoretical arguments [27], which seem more convincing, tell that virtually all $KK\pi$ production takes place in the axial channel. Therefore, in fitting the data, I subtract the $KK\pi$ yield from the data points presented in Fig. 10. I hasten to add that the subtraction affects only the energy range $s > 2 \text{ GeV}^2$, i.e. the second peak in Fig. 10, and even in this energy range the effect is small, $\lesssim 5\%$. The subtraction is not essential for a general picture I draw here.

The solid curve at $s < 2.7 \text{ GeV}^2$ is the best fit of the data points thus obtained by the sum of two Breit-Wigner peaks (the first one, the ρ meson, is a modified Breit-Wigner taking into account threshold effects important for the ρ meson). For my purposes the two-resonance fit presents an excellent approximation to the experimental spectral density at $s < 2.7 \text{ GeV}^2$.

The tails of the curves at $2.7 < s < 7 \text{ GeV}^2$ represent the resonance-based and instanton-based models

(the solid and dashed curves, respectively) which I discussed in Secs. 10 and 6. The solid curve is a modulation of $\rho(s)_{\text{pert}}$ by the factor

$$1 + 1.22 E^{-3} \sin(2\rho E - \delta),$$

$$\rho = 3 \text{ GeV}^{-1}, \quad \delta = 1.32, \quad E \text{ in GeV},$$

cf. Eq. (18). The dashed curve is a modulation of $\rho(s)_{\text{pert}}$ by⁶

$$1 - 1.24 \exp\left(-\frac{2\pi s B}{\sigma^2 N_c}\right) \sin\left(\frac{2\pi s}{\sigma^2} - 3.08\right),$$

$$\sigma^2 = 2 \text{ GeV}^2, \quad B = 0.5, \quad N_c = 3,$$

cf. Eq. (58). Finally, the thick solid curve in Fig. 10 displays $\rho(s)_{\text{pert}}$ calculated through order α_s^3 , with $\Lambda_{\overline{\text{MS}}}^{(3)} = 200 \text{ MeV}$.

It is clearly seen that, qualitatively, both models match the data well. In fact, in actuality I would expect an oscillating component which is a combination of the solid and dashed curves, since the mechanisms of the duality violations they represent are complementary. The instanton mechanism leads to a visibly slower (power) fall off, with rarer oscillations. It is also clear that experimental measurements at the percent level of accuracy will most certainly provide us with the material needed for construction of a reliable and well calibrated model of the duality violations.

12 Conclusions

I identified general mechanisms causing deviations from duality and derived scaling laws governing the damping of the duality violating component in a variety of inclusive processes as a function of energy (momentum transfer). The estimates of the absolute normalization one can perform at the moment are less certain. This explains why I was so cautious in my discussion of the numerical situation in R_τ , and completely avoided this issue in other inclusive processes. Getting a better idea of the absolute normalization is absolutely necessary for all practical applications of the theoretical constructions presented here. Precision measurements of $\rho(s)$ in a wide energy range would be of enormous help in this question and would probably lead to a speedy solution.

As I have already noted, duality violations parametrize our ignorance. Were an analytic solution of QCD found, the contents of this review would

⁶As we know from Sec. 10, the equidistant resonances with equal residues result in oscillations of this type superimposed on $\rho = 1$. By a trivial adjustment of the residues one can readily achieve that the oscillations (58) are superimposed on $\rho(s)_{\text{pert}}$.

become instantly obsolete. Then the exact asymptotic behavior of inclusive cross sections would be known, and the very concept of duality violations would become irrelevant.

Acknowledgments

I am deeply grateful to Pietro Colangelo, Fulvia De Fazio, and Giuseppe Nardulli for the kind hospitality extended to me in Conversano.

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