Bounds on the derivatives of the Isgur-Wise function

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Using the OPE, we formulate new sum rules in the heavy quark limit of QCD. These sum rules imply that the elastic Isgur-Wise function $\xi(w)$ is an alternate series in powers of $(w-1)$. Moreover, one gets that the n-th derivative of $\xi(w)$ at $w=1$ can be bounded by the $(n-1)$-th one, and an absolute lower bound for the n-th derivative $(-1)^n\xi^{(n)}(1) \geq \frac{(2n+1)!!}{2^{2n}}$. Moreover, for the curvature we find $\xi''(1) \geq \frac{1}{5}[4\rho^2 + 3(\rho^2)^2]$ where $\rho^2 = -\xi'(1)$. We show that the quadratic term $\frac{3}{5}(\rho^2)^2$ has a transparent physical interpretation, as it is leading in a non-relativistic expansion in the mass of the light quark. These bounds should be taken into account in the parametrizations of $\xi(w)$ used to extract $|V_{cb}|$. These results are consistent with the dispersive bounds, and they strongly reduce the allowed region of the latter for $\xi(w)$.

In the leading order of the heavy quark expansion of QCD, Bjorken sum rule (SR) [1] relates the slope of the elastic Isgur-Wise (IW) function $\xi(w)$, to the IW functions of the transitions between the ground state and the $j^P = \frac{1}{2}^+, \frac{3}{2}^+$ excited states, $\tau_{1/2}^{(n)}(w)$, $\tau_{3/2}^{(n)}(w)$, at zero recoil $w=1$ (n is a radial quantum number). This SR leads to the lower bound $-\xi'(1) = \rho^2 \geq \frac{1}{4}$. Recently, a new SR was formulated by Uraltsev in the heavy quark limit [2] involving also $\tau_{1/2}^{(n)}(1)$, $\tau_{3/2}^{(n)}(1)$, that implies, combined with Bjorken SR, the much stronger lower bound $\rho^2 \geq \frac{3}{4}$, a result that came as a big surprise. In ref. [3], in order to make a systematic study in the heavy quark limit of QCD, we have developed a manifestly covariant formalism within the Operator Product Expansion (OPE). We did recover Uraltsev SR plus a new class of SR. Making a natural physical assumption, this new class of SR imply the bound $\sigma^2 \geq \frac{5}{4}\rho^2$ where σ^2 is the curvature of the IW function. Using this formalism including the whole tower of excited states j^P, we have recovered rigorously the bound $\sigma^2 \geq \frac{5}{4}\rho^2$ plus generalizations that extend it to all the derivatives of the IW function $\xi(w)$ at zero recoil, that is shown to be an alternate series in powers of $(w-1)$.

Using the OPE and the trace formalism in the heavy quark limit, different initial and final four-velocities v_i and v_f, and heavy quark currents, where Γ_1 and Γ_2 are arbitrary Dirac matrices $J_1 = \bar{h}_{v'}^{(c)}\ \Gamma_1\ h_{v_i}^{(b)}$, $J_2 = \bar{h}_{v_f}^{(b)}\ \Gamma_2\ h_{v'}^{(c)}$, the following sum rule can be written [4] :

$$\Big\{ \sum_{D=P,V} \sum_n Tr\left[\bar{\mathcal{B}}_f(v_f)\bar{\Gamma}_2\mathcal{D}^{(n)}(v')\right] Tr\left[\bar{\mathcal{D}}^{(n)}(v')\Gamma_1\mathcal{B}_i(v_i)\right]\xi^{(n)}(w_i)\xi^{(n)}(w_f) + \text{Other excited states}\Big\} = -2\xi(w_{if})\, Tr\left[\bar{\mathcal{B}}_f(v_f)\bar{\Gamma}_2P'_+\Gamma_1\mathcal{B}_i(v_i)\right] \ . \qquad (1)$$

In this formula v' is the intermediate meson four-velocity, $P'_+ = \frac{1}{2}(1+\not{v}')$ comes from the residue of the positive energy part of the c-quark propagator, $\xi(w_{if})$ is the elastic Isgur-Wise function that appears because one assumes $v_i \neq v_f$. $\mathcal{B}_i$ and $\mathcal{B}_f$ are the 4×4 matrices of the ground state B or B^* mesons and $\mathcal{D}^{(n)}$ those of all possible ground state or excited state D mesons coupled to B_i and B_f through the currents. In (1) we have made explicit the $j=\frac{1}{2}^-$ D and D^* mesons and their radial excitations of quantum number n. The explicit contribution of the other excited states is written below.

The variables w_i, w_f and w_{if} are defined as $w_i = v_i\cdot v'$, $w_f = v_f\cdot v'$, $w_{if} = v_i\cdot v_f$.

The domain of $(w_i,\, w_f,\, w_{if})$ is [3] $(w_i, w_f \geq 1)$

$$w_iw_f - \sqrt{(w_i^2-1)(w_f^2-1)} \leq w_{if} \leq w_iw_f + \sqrt{(w_i^2-1)(w_f^2-1)}\ . \qquad (2)$$

The SR (1) writes $L(w_i, w_f, w_{if}) = R(w_i, w_f, w_{if})$, where $L(w_i, w_f, w_{if})$ is the sum over the intermediate charmed states and $R(w_i, w_f, w_{if})$ is the OPE side. Within the domain (2) one can derive relatively to any of the variables w_i, w_f and w_{if} and obtain different SR taking different limits to the frontiers of the domain.

As in ref. [3], we choose as initial and final states the B meson $\mathcal{B}_i(v_i) = P_{i+}(-\gamma_5)$ $\mathcal{B}_f(v_f) = P_{f+}(-\gamma_5)$ and vector or axial currents projected along the v_i and v_f four-velocities

$$J_1 = \bar{h}^{(c)}_{v'}\ \not{v}_i\ h^{(b)}_{v_i} \quad , \qquad J_2 = \bar{h}^{(b)}_{v_f}\ \not{v}_f\ h^{(c)}_{v'} \tag{3}$$

we obtain SR (1) with the sum of all excited states j^P in a compact form :

$$\begin{aligned}&(w_i+1)(w_f+1)\sum_{\ell\geq 0}\frac{\ell+1}{2\ell+1}S_\ell(w_i,w_f,w_{if})\\ &\sum_n \tau^{(\ell)(n)}_{\ell+1/2}(w_i)\tau^{(\ell)(n)}_{\ell+1/2}(w_f)\\ &+\sum_{\ell\geq 1}S_\ell(w_i,w_f,w_{if})\sum_n \tau^{(\ell)(n)}_{\ell-1/2}(w_i)\tau^{(\ell)(n)}_{\ell-1/2}(w_f)\\ &=(1+w_i+w_f+w_{if})\xi(w_{if})\ .\end{aligned} \tag{4}$$

We get, choosing instead the axial currents,

$$J_1 = \bar{h}^{(c)}_{v'}\ \not{v}_i\ \gamma_5\ h^{(b)}_{v_i} \quad , \qquad J_2 = \bar{h}^{(b)}_{v_f}\ \not{v}_f\ \gamma_5\ h^{(c)}_{v'}\ , \tag{5}$$

$$\begin{aligned}&\sum_{\ell\geq 0}S_{\ell+1}(w_i,w_f,w_{if})\sum_n \tau^{(\ell)(n)}_{\ell+1/2}(w_i)\tau^{(\ell)(n)}_{\ell+1/2}(w_f)\\ &+(w_i-1)(w_f-1)\sum_{\ell\geq 1}\frac{\ell}{2\ell-1}S_{\ell-1}(w_i,w_f,w_{if})\\ &\sum_n \tau^{(\ell)(n)}_{\ell-1/2}(w_i)\tau^{(\ell)(n)}_{\ell-1/2}(w_f)\\ &=-(1-w_i-w_f+w_{if})\xi(w_{if})\ .\end{aligned} \tag{6}$$

Following the formulation of heavy-light states for arbitrary j^P given by Falk [4], we have defined in ref. [3] the IW functions $\tau^{(\ell)(n)}_{\ell+1/2}(w)$ and $\tau^{(\ell)(n)}_{\ell-1/2}(w)$, ℓ and $j=\ell\pm\frac{1}{2}$ being the orbital and total angular momentum of the light quark.

In equations (3) and (5) the quantity S_n is

$$S_n = v_{i\nu_1}\cdots v_{i\nu_n}\ v_{f\mu_1}\cdots v_{f\mu_n}\ T^{\nu_1\cdots\nu_n,\mu_1\cdots\mu_n} \tag{7}$$

and the polarisation projector $T^{\nu_1\cdots\nu_n,\mu_1\cdots\mu_n}$

$$T^{\nu_1\cdots\nu_n,\mu_1\cdots\mu_n} = \sum_\lambda \varepsilon'^{(\lambda)*\nu_1\cdots\nu_n}\ \varepsilon'^{(\lambda)\mu_1\cdots\mu_n} \tag{8}$$

depends only on the four-velocity v'. The polarisation tensor $\varepsilon'^{(\lambda)\mu_1\cdots\mu_n}$ is a traceless symmetric tensor, a symmetric tensor with vanishing contractions, transverse relatively to v'. Moreover [3] :

$$\begin{aligned}S_n = &\sum_{0\leq k\leq \frac{n}{2}} C_{n,k}(w_i^2-1)^k(w_f^2-1)^k\\ &(w_iw_f-w_{if})^{n-2k}\end{aligned} \tag{9}$$

with $C_{n,k}=(-1)^k\frac{(n!)^2}{(2n)!}\ \frac{(2n-2k)!}{k!(n-k)!(n-2k)!}$.

From the sum of (4) and (6) one obtains, differentiating relatively to w_{if} [5] $(\ell\geq 0)$:

$$\begin{aligned}\xi^{(\ell)}(1) = &\frac{1}{4}\ (-1)^\ell\ \ell!\left\{\frac{\ell+1}{2\ell+1}4\sum_n\left[\tau^{(\ell)(n)}_{\ell+1/2}(1)\right]^2\right.\\ &\left.+\sum_n\left[\tau^{(\ell-1)(n)}_{\ell-1/2}(1)\right]^2+\sum_n\left[\tau^{(\ell)(n)}_{\ell-1/2}(1)\right]^2\right\}\ .\end{aligned} \tag{10}$$

This relation shows that $\xi(w)$ is an alternate series in powers of $(w-1)$. Equation (10) reduces to Bjorken SR [1] for $\ell=1$. Differentiating (6) relatively to w_{if} and making $w_i=w_f=w_{if}=1$ one obtains :

$$\xi^{(\ell)}(1) = \ell!\ (-1)^\ell\sum_n\left[\tau^{(\ell)(n)}_{\ell+1/2}(1)\right]^2 \quad (\ell\geq 0)\ . \tag{11}$$

Combining (10) and (11) one obtains :

$$\begin{aligned}&\frac{\ell}{2\ell+1}\sum_n\left[\tau^{(\ell)(n)}_{\ell+1/2}(1)\right]^2-\frac{1}{4}\sum_n\left[\tau^{(\ell)(n)}_{\ell-1/2}(1)\right]^2\\ &=\frac{1}{4}\sum_n\left[\tau^{(\ell-1)(n)}_{\ell-1/2}(1)\right]^2\end{aligned} \tag{12}$$

that reduces to Uraltsev SR [2] for $\ell=1$. From (10) and (11) one obtains :

$$\begin{aligned}&(-1)^\ell\ \xi^{(\ell)}(1) = \frac{1}{4}\ \frac{2\ell+1}{\ell}\ell!\\ &\left\{\sum_n\left[\tau^{(\ell-1)(n)}_{\ell-1/2}(1)\right]^2+\sum_n\left[\tau^{(\ell)(n)}_{\ell-1/2}(1)\right]^2\right\}\ .\end{aligned} \tag{13}$$

implying

$$(-1)^\ell\xi^{(\ell)}(1)\geq\frac{2\ell+1}{4}\left[(-1)^{\ell-1}\xi^{(\ell-1)}(1)\right] \tag{14}$$

and the absolute bound

$$(-1)^\ell\xi^{(\ell)}(1)\geq\frac{(2\ell+1)!!}{2^{2\ell}} \tag{15}$$

that gives, in particular, for the lower cases,

$$-\xi'(1)=\rho^2\geq\frac{3}{4} \quad , \quad \xi''(1)\geq\frac{15}{16} \tag{16}$$

Let us first consider the derivatives of the SR for vector currents (4) relatively to w_{if} with the boundary condition $w_{if}=1$. For $w_{if}=1$, the domain (2) implies : $w_i=w_f=w$. We define therefore $L_V(w_{if},w)\equiv$

$L_V(w_{if}, w_i, w_f)|_{w_i=w_f=w}$, and likewise for R_V. We then take the $p+q$ derivatives

$$\left(\frac{\partial^{p+q} L_V}{\partial w_{if}^p \partial w^q}\right)_{w_{if}=w=1} = \left(\frac{\partial^{p+q} R_V}{\partial w_{if}^p \partial w^q}\right)_{w_{if}=w=1} \quad (17)$$

and exploit systematically the obtained relations. To get information on the curvature σ^2 of the elastic IW function we go to the second order derivatives.

Let us consider likewise the derivatives of the SR for axial currents (6) with the boundary condition $w_{if} = 1$, $w_i = w_f = w \to 1$ like in (17). We obtain :

$$\rho^2 - \frac{4}{5}\sigma^2 + \sum_n |\tau_{3/2}^{(2)(n)}(1)|^2 = 0 \ . \quad (18)$$

$$\frac{4}{3}\rho^2 + (\rho^2)^2 - \frac{5}{3}\sigma^2 + \sum_{n\neq 0} |\xi^{(n)'}(1)|^2 = 0 \quad (19)$$

that imply :

$$\sigma^2 \geq \frac{5}{4}\ \rho^2 \quad (20)$$

$$\sigma^2 \geq \frac{1}{5}\left[4\rho^2 + 3(\rho^2)^2\right] \ . \quad (21)$$

There is a simple intuitive argument to understand the term $\frac{3}{5}(\rho^2)^2$ in the best bound (21). Let us consider the non-relativistic quark model, i.e. a non-relativistic light quark q interacting with a heavy quark Q through a potential. The form factor – to be identified with the IW function – has then the simple form :

$$F(\mathbf{k}^2) = \int d\mathbf{r}\ \varphi_0^+(r) \exp\left(i\frac{m_q}{m_q+m_Q}\mathbf{k}\cdot\mathbf{r}\right)\varphi_0(r) \quad (22)$$

where $\varphi_0(r)$ is the ground state radial wave function. Identifying the non-relativistic IW function $\xi_{NR}(w)$ with the form factor $F(\mathbf{k}^2)$ (22), one finds, because of rotational invariance :

$$\xi_{NR}(w) \cong 1 - m_q^2 < 0|z^2|0 > (w-1)$$
$$+\frac{1}{2}\ \frac{1}{3}\ m_q^4 < 0|z^4|0 > (w-1)^2 + \cdots \quad (23)$$

where $|0>$ stands for the ground state wave function. Therefore, one has the following expressions for the slope and the curvature, in the non-relativistic limit :

$$\rho^2_{NR} = m_q^2 < 0|z^2|0 > \ , \ \sigma^2_{NR} = \frac{1}{3}\ m_q^4 < 0|z^4|0 > \ . \quad (24)$$

From spherical symmetry and completeness, one can prove then,

$$\sigma^2_{NR} \geq \frac{3}{5}\ \left[\rho^2_{NR}\right]^2 \ . \quad (25)$$

Notice that, denoting by R the bound state radius and m_q the light quark mass, in the non-relativistic limit, $(\rho^2_{NR})^2$ and σ^2_{NR} scale like $m_q^4 R^4$.

An interesting phenomenological remark is that the simple parametrization for the IW function [6]

$$\xi(w) = \left(\frac{2}{w+1}\right)^{2\rho^2} \quad (26)$$

satisfies the inequalities (14), (20)-(21) if $\rho^2 \geq \frac{3}{4}$.

The result (15), that shows that all derivatives at zero recoil are large, should have important phenomenological implications for the empirical fit needed for the extraction of $|V_{cb}|$ in $B \to D^*\ell\nu$. The usual fits to extract $|V_{cb}|$ using a linear or linear plus quadratic dependence of $\xi(w)$ are not accurate enough.

As a simple example of a fit with the simple function (26), we can use BELLE data on $\bar{B}^0 \to D^{*+}e^-\bar{\nu}$ for the product $|V_{cb}|\mathcal{F}^*(w)$ [7]. The function $\mathcal{F}^*(w)$ is equal to the Isgur-Wise function $\xi(w)$ in the heavy quark limit. Assuming only departures of this limit at $w = 1$, i.e. we fit $\xi(w)$ from the data with $|V_{cb}|\mathcal{F}^*(w) = |V_{cb}|\mathcal{F}^*(1)\xi(w)$, we obtain the following results for the normalization and the slope $\mathcal{F}^*(1)|V_{cb}| = 0.036 \pm 0.002$, $\rho^2 = 1.15 \pm 0.18$.

As we can see, the determination of $\mathcal{F}^*(1)|V_{cb}|$ is rather precise, while ρ^2 has a larger error. However, the values obtained for $|V_{cb}|$ and ρ^2 are strongly correlated. It is important to point out that the most precise data points are the ones at large w, so that higher derivatives contribute importantly in this region. Due to the alternate character of $\xi(w)$ as a series of $(w-1)$, one does not see strongly the curvature of $\xi(w)$, but the curve is definitely not close to a straight line.

A considerable effort has been developed to formulate dispersive constraints on the shape of the form factors in $\bar{B} \to D^*\ell\nu$ [8]-[9]. The starting point are the analyticity properties of two-point functions and positivity of the corresponding spectral functions. Dispersion relations relate the hadronic spectral functions to the QCD two-point functions in the deep Euclidean region, and positivity allows to bounds the contribution of the relevant states, leading to constraints on the semileptonic form factors.

Our approach, based on Bjorken-like SR, holds *in the physical region* of the semileptonic decays $\bar{B} \to D^{(*)}\ell\nu$

and *in the heavy quark limit*. Concerning this last simplifying feature, we should underline that there is no objection of principle to include in the calculation radiative corrections and subleading corrections in powers of $1/m_Q$.

The dispersive approach starts from bounds *in the crossed channel* by comparison of the OPE and the sum over hadrons coupled to the corresponding current, $\bar{B}\bar{D}$, $\bar{B}\bar{D}^*$, $\cdots$ and one analytically continues to the physical region of the semileptonic decays. This is done for a single reference form factor, for example the combination $V_1(w) = h_+(w) - \frac{m_B - m_D}{m_B + m_D}\, h_-(w)$ that enters in the $\bar{B} \to D\ell\nu$ rate. Ratios of the remaining form factors to $V_1(w)$ are computed *in the physical region* by introducing $1/m_Q$ and α_s corrections to the heavy quark limit. The dispersive approach considers *physical quark masses*, in contrast with the heavy quark limit of our method.

The two approaches are quite different in spirit and in their results. However, it can be interesting to compare numerically our bounds with the ones of the dispersive approach, as they happen to be complementary. We must however keep in mind precisely the differences between the two methods.

We have demonstrated in [5] that the IW function $\xi(w)$ is an alternating series in powers of $(w-1)$, with the moduli of the derivatives satisfying the bounds (14) and (21).

Let us consider the main results of ref. [9], that are summarized by the one-parameter formula

$$\frac{V_1(w)}{V_1(1)} \cong 1 - 8\rho^2 z + (51\rho^2 - 10)z^2 - (252\rho^2 - 84)z^3 \quad (27)$$

with the variable $z(w)$ defined by

$$z = \frac{\sqrt{w+1} - \sqrt{2}}{\sqrt{w+1} + \sqrt{2}} \quad (28)$$

and the allowed range for ρ^2 being

$$-0.17 < \rho^2 < 1.51\ . \quad (29)$$

Of course, the function $\frac{V_1(w)}{V_1(1)}$ contains finite mass corrections that are absent at present in our method. Nevertheless, let us compare these results with our lower bounds, assuming the rough approximation

$$\frac{V_1(w)}{V_1(1)} \cong \xi(w)\ . \quad (30)$$

Let us now comment on the implications of our type of bounds. The simplest important remark is that, within the simplifying hypothesis (30), (29) is considerably tightened by the lower bound on $\rho^2 \geq \frac{3}{4}$:

$$\frac{3}{4} \leq \rho^2 < 1.51 \quad (31)$$

that shows that our lower bounds are complementary to the upper bounds obtained from dispersive methods. Within the hypothesis of the heavy quark limit, the region allowed by the dispersive bounds for $\xi(w)$ with ρ^2 within the range (29) is much reduced by the bounds (31).

In conclusion, using sum rules in the heavy quark limit of QCD, as formulated in ref. [3], we have found lower bounds for the moduli of the derivatives of $\xi(w)$. Any phenomenological parametrization of $\xi(w)$ intending to fit the CKM matrix element $|V_{cb}|$ in $B \to D^{(*)}\ell\nu$ should satisfy these bounds. Moreover, we discuss these bounds in comparison with the dispersive approach. We show that there is no contradiction, our bounds restraining the region for $\xi(w)$ allowed by this latter method.

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