Theory of the Muon Anomalous Magnetic Moment

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Standard Model prediction for the muon anomalous magnetic moment $(g-2)$ is reviewed. Recent shifts in the QED and hadronic contributions are discussed. The result is compared with the latest Brookhaven E821 measurement.

1. Introduction

In February 2001 the Brookhaven collaboration E821 announced a new measurement of the muon anomalous magnetic moment [1],

$$a_\mu = 116\ 592\ 020(160) \times 10^{-11} \tag{1}$$

which exceeded the theoretical prediction by about 2.6 standard deviations. This disagreement motivated a large number of theoretical speculations about various New Physics scenarios. The Standard Model (SM) prediction has also been scrutinized and, eventually, found to have been flawed. The difference between theory and experiment has been reduced almost to 1σ.

More recently [2], E821 announced a new result, based on an analysis of all data taken with positive muons. The resulting world average is

$$a_\mu^{\text{exp}}(\text{Average}) = 116\,592\,030(80) \times 10^{-11}. \tag{2}$$

Due to the greatly reduced experimental error, and the continuing progress in analyzing e^+e^- annihilation and τ decay data, essential for the theoretical prediction, this result again differs significantly from the SM prediction.

The SM prediction for a_μ is given by a sum of QED, hadronic, and electroweak contributions,

$$a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{Had}} + a_\mu^{\text{EW}}. \tag{3}$$

These three terms have been discussed in detail in recent reviews [3,4]. Since those studies, the hadronic part, and to a lesser extent also the QED and electroweak ones, have changed. Those developments are summarized in subsequent sections.

2. Standard Model contributions

2.1. QED

QED contribution is by far the largest part of the muon anomalous magnetic moment. It is expressed by a truncated expansion in $\alpha \simeq 137.036$,

$$a_\mu^{\text{QED}} = \sum_{n=1}^{5} C_n \left(\frac{\alpha}{\pi}\right)^n. \tag{4}$$

The first three coefficients C_i are known analytically (see [5] and references therein). C_4 has been computed numerically and its various elements are still being checked and improved. It is now believed to exceed by several units the preliminary value $C_4^{\text{old}} = 126.07(41)$ [6]. This shifts the value of a_μ^{QED} by about $+15 \times 10^{-11}$ or five times the theoretical uncertainty usually assigned to the QED part. An independent evaluation of C_4 would certainly be very helpful.

The five-loop contribution C_5 is estimated by studying diagrams most strongly enhanced by logarithms of the muon and electron mass ratio. It amounts to about 6×10^{-11} in a_μ, not very important for the E821 accuracy goal of 40×10^{-11}.

The uncertainty of $+3 \times 10^{-11}$ in a_μ^{QED} arises in roughly equal measure from errors assigned to α and $C_{4,5}$, and a small number estimating the higher order terms in the QED series. However, until an updated numerical result for a_μ^{QED}, expected in the near future [7], is published, it may be safe to increase the error estimate to about $+15 \times 10^{-11}$:

$$a_\mu^{\text{QED}} = 116\,584\,721(15) \times 10^{-11}. \tag{5}$$

2.2. Electroweak contribution

Observation of the electroweak loops was among the original goals of E821. This is the smallest of the SM contributions and the only one not seen in the earlier CERN experiment.

The leading electroweak effect arises from one-loop diagrams with W and Z bosons. Two-loop contributions decrease it by about 23%, due to large logarithms of the muon and weak boson mass ratio [8–11]. Also the three-loop leading logarithms have been evaluated [12,13] but their effect is negligible.

Recently, there has been some controversy regarding the treatment of light-quark loops in two-loop electroweak diagrams, such as the one shown in Fig. 1.

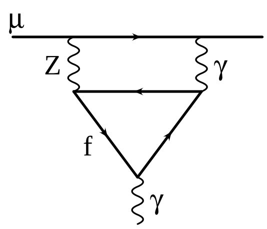

Figure 1. Effective $Z\gamma\gamma^*$ coupling induced by a fermion triangle, contributing to $a_\mu^{\rm EW}$.

Such diagrams give rise to large logarithms of the ratio of the Z-boson mass (M_Z) and the muon or light-hadron mass. It was found in [10] that when one adds triangle loop contributions with all charged fermions in a given generation, those logarithms cancel. More recently, in [14] it was claimed that this cancellation is spurious and due to a simplistic treatment of light hadrons in [10]. However, it seems unlikely that a low-energy effect of strong interactions can alter the ultraviolet asymptotics of such diagrams, reflected in the coefficient of the leading logarithm of M_Z. This issue was reexamined in [12]. The conclusion of that study is that the operator product expansion analysis in [14] is flawed; part of the short-distance contribution to the virtual triangle in Fig. 1 is missing in [14]. When this effect is added, the cancellation of the large logarithms is restored.

Somewhat surprisingly, the numerical result is very similar in both approaches. It depends weakly on the Higgs mass m_H, and for $m_H = 150$ GeV is

$$a_\mu^{\rm EW} = 154(1)(2) \times 10^{-11}. \qquad (6)$$

where the first error corresponds to hadronic loop uncertainties and the second to an allowed Higgs mass range of 114 GeV $\lesssim m_H \lesssim$ 250 GeV, the current top mass uncertainty and unknown three-loop effects.

2.3. Hadronic effects

Hadronic contributions to $g-2$ are usually divided into three parts. The largest contribution is the leading order vacuum polarization (LOVP). Verification and further reduction of its uncertainty are crucial for extracting interesting physics information from the E821 measurement. Some details of the LOVP calculation are discussed below.

Second, there is the next-to-leading order vacuum polarization (NLVP) contribution, of about $-100(6) \times 10^{-11}$. It is obtained with a similar procedure as the LOVP but is suppressed by an extra factor α/π. Its present accuracy is sufficient for E821 purposes.

The third part arises due to the hadronic light-by-light (HLBL) scattering. This is also a relatively small contribution, $110(30) \times 10^{-11}$, but its uncertainty and even the central value are still somewhat controversial. This will be discussed at the end of this section.

2.3.1. Vacuum polarization

The largest part of hadronic contributions to a_μ comes from a vacuum polarization insertion into the one-loop QED diagram. By applying dispersion relation to the photon vacuum polarization, this diagram can be rewritten as a convolution of a kernel function and the e^+e^- annihilation cross section into hadrons. Since the kernel function falls off for large values of the integration variable s in dispersion integral, running from $4m_\pi^2$ to ∞, the integral is saturated at $\sqrt{s} \leq 2$ GeV. Since the cross section of e^+e^- annihilation into hadrons cannot be computed from first principles at such low energies, one has to rely on the experimental data. One can use the data on $e^+e^- \to$ hadrons or employ the data on decays $\tau \to \nu_\tau +$ hadrons and, using isospin symmetry, relate it to $e^+e^- \to$ hadrons.

Both methods have been applied in the past and each of them has different merits and shortcomings. e^+e^- data have the advantage of being exactly what enters the dispersion integral, although some theoretical work, like removal of the initial state radiation and vacuum polarization corrections, has to be applied. The disadvantage of this method is that for a long time the accuracy of e^+e^- data below 1.7 GeV was insufficient for the precision of E821. This has changed recently. New, very precise data became available on $e^+e^- \to \pi^+\pi^-$ in the ρ resonance region and new results on $\sigma(e^+e^- \to$ hadrons) at 2 GeV $\leq \sqrt{s} \leq$ 3 GeV have been obtained.

The use of τ decay data has been largely motivated by the accuracy of ALEPH and CLEO measurements. However, the disadvantage of this method is the amount of theoretical assumptions one has to make in order to get from τ decays to e^+e^- annihilation cross sections. For example, a convincing study of isospin violating effects (electromagnetism, quark masses, $\rho - \omega$ interference) is required before the precision of 1% can be guaranteed.

A recent study [15] presents two separate results for the vacuum polarization contribution. If only e^+e^- data are employed, the result reads $6847(70) \times 10^{-11}$. Results from τ decays are more accurate in the energy region where they are available, but unfortunately they significantly differ from e^+e^- data. The tau-aided result is $7090(59) \times 10^{-11}$, and as we will see it leads to a very good agreement of the Standard Model with the $g-2$ measurement.

Another study of e^+e^- data [16] found the value $6831(62) \times 10^{-11}$, confirming the above e^+e^- result.

2.3.2. Light-by-light scattering

The hadronic light-by-light (HLBL) scattering contribution is the trickiest part of the theoretical prediction for a_μ because (a) the typical loop momenta are of the order of 1 GeV or less (see Fig. 2 for an example of hadronic contributions), (b) it seems impossible to relate this contribution to experimental data. Assuming that very small momentum transfers $k \sim m_\mu \sim m_\pi$ saturate HLBL, one might attempt to use the chiral perturbation theory to estimate a_μ(HLBL). Unfortunately, this basic assumption about the smallness of momenta is not valid for HLBL scattering diagrams and in fact the Feynman integrals are saturated at a relatively high scale $k \sim m_\rho \sim 1$ GeV, where the arguments based on χPT alone are insufficient. Within the χPT alone one cannot determine the UV counterterm proportional to the muon anomalous magnetic moment. For this reason one has to rely on models (vector meson dominance, Nambu-Jona Lasinio) to describe contributions of large momentum degrees of freedom.

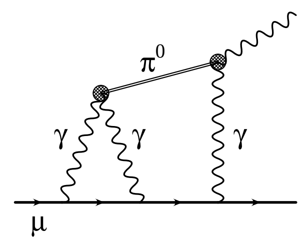

Figure 2. Pion pole contribution to hadronic light-by-light effect in the muon $g-2$.

Since it is obviously quite difficult to evaluate reliability of these models and to estimate convincingly the theoretical uncertainty of their predictions, the final result for HLBL contribution to $g-2$ and the estimate of its uncertainty are very subjective and differ among recent studies [17–19]. The final result here is based on the VMD model for π^0 [20], and charged pion contributions [21]; the missing high-energy part of HLBL (the counterterm) is estimated using the quark loop diagram with an infrared cut-off provided by the quark mass $M_Q = 200 - 400$ MeV. The result for HLBL is then $a_\mu(\mathrm{HLBL}) = 110(30) \times 10^{-11}$ (this includes a preliminary re-examination of the pion box diagrams).

Given the discrepancy between the electron-positron annihilation and tau decay results, we are left with two SM predictions. They differ from each other by about 2.3 times the errors combined in quadrature,

$$a_\mu^{\rm Had} = \begin{cases} 6857(76) \times 10^{-11} & e^+e^-\text{–based} \\ 7100(67) \times 10^{-11} & \tau\text{–aided} \end{cases} \quad (7)$$

3. Summary

The complete Standard Model prediction is obtained by adding equations (5,6,7),

$$a_\mu^{\rm SM} = \begin{cases} 116\,591\,732(78) \times 10^{-11} & e^+e^-\text{–based} \\ 116\,591\,975(69) \times 10^{-11} & \tau\text{–aided} \end{cases} \quad (8)$$

Present experimental world average, eq. (2), exceeds both theoretical numbers,

$$a_\mu^{\rm exp}(\text{Average}) - \mathrm{a}_\mu^{\rm SM} = \begin{cases} 298(112) \times 10^{-11} & (2.7\sigma) & e^+e^-\text{–based} \\ 55(106) \times 10^{-11} & (0.5\sigma) & \tau\text{–aided} \end{cases} \quad (9)$$

The experimental error will be further reduced when data taken with negative muons are analyzed. It is very important to reduce theoretical errors in the hadronic contributions to match those improvements. As far as the vacuum polarization contribution is concerned, we can soon expect an independent check of the e^+e^- results based on radiative-return data obtained at DAΦNE and at B-factories. A new analysis of τ decays will likely be necessary to clarify the discrepancy between the two approaches. Such analysis may be based on the very large sample of τ data collected at B-factories.

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