

# The $\tau^+\tau^-$ production cross section near threshold revisited

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Next-to-next-to-leading contributions to the cross section  $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$  at energies close to threshold are analysed, taking into account the known non-relativistic effects and  $\mathcal{O}(\alpha^2)$  corrections. The numerical changes with respect to previous works are small, but the new corrections give a true estimate of the uncertainty in the theoretical calculation.

## 1. Introduction

The Tau-Charm Factory, a high-luminosity ( $\sim 10^{33} \text{ cm}^{-2} \text{ s}^{-1}$ )  $e^+e^-$  collider with a centre-of-mass energy near the  $\tau^+\tau^-$  production threshold, has been proposed as a powerful tool to perform high-precision studies of the  $\tau$  lepton, charm hadrons and the charmonium system [1]. In recent years, this energy region has been only partially explored by the Chinese BEBC machine ( $\sim 10^{31} \text{ cm}^{-2} \text{ s}^{-1}$ ). The possibility to operate the Cornell CESR collider around the  $\tau^+\tau^-$  threshold [2] has revived again the interest on Tau-Charm Factory physics. Also the CMD-2 detector at the Novosibirsk VEPP-2M  $e^+e^-$  collider will be soon ready to collect new data in this region [3].

A precise understanding of the  $e^+e^- \rightarrow \tau^+\tau^-$  production cross section near threshold is clearly required. The accurate experimental analysis of this observable could allow to improve the present measurement [4] of the  $\tau$  lepton mass. The cross section  $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$  has already been analysed to  $\mathcal{O}(\alpha^3)$  in refs. [5], including a resummation of the leading Coulomb corrections.

The recent development of non-relativistic effective field theories of QED (NRQED) and QCD (NRQCD) [6] has allowed an extensive investigation of the threshold production of heavy flavours at  $e^+e^-$  colliders. The threshold  $b\bar{b}$  [7] and  $t\bar{t}$  [8] production cross sections have been computed to the next-to-next-to-leading order (NNLO) in a combined expansion in powers of  $\alpha_s$  and the fermion velocities. Making appropriate changes, those calculations can be easily applied to the

study of  $\tau^+\tau^-$  production [9]. One can then achieve a theoretical precision better than 0.1%.

## 2. Perturbative calculation to $\mathcal{O}(\alpha^4)$

At lowest order in QED, the  $\tau$  leptons are produced by one-photon exchange in the s-channel, and the total cross section formula reads

$$\sigma_B(e^+e^- \rightarrow \tau^+\tau^-) = \frac{2\pi\alpha^2}{3s} v(3-v^2), \quad (1)$$

where  $v = \sqrt{1 - 4m_\tau^2/s}$  is the velocity of the final  $\tau$  leptons in the center-of-mass frame of the  $e^+e^-$  pair which makes  $\sigma_B$  vanish when  $v \rightarrow 0$ .

Electromagnetic corrections of  $\mathcal{O}(\alpha)$  arise from the interference between the tree level result and 1-loop amplitudes. A factor  $\alpha/v$  emerges in the 1-loop final state interaction between the tau leptons, making the cross section at threshold finite. Furry's theorem guarantees that contributions to  $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$  coming from initial, intermediate and final state corrections completely factorize at  $\mathcal{O}(\alpha^3)$ , including real photon emission.

Some undesirable features appear at  $\mathcal{O}(\alpha^4)$ : The two-loop  $\tau^+\tau^-\gamma$  vertex develops an  $\alpha^2/v^2$  term which makes the cross section ill-defined when  $v \rightarrow 0$ , and multiple photon production of tau leptons by box-type diagrams and the non-zero interference of initial and final state radiation spoil exact factorization. However, as it has been recently shown [10], the squared amplitude of the  $e^+e^- \rightarrow \tau^+\tau^-$  box diagram (see Fig. 1) is proportional to  $\alpha^4 v^2$ , and so represents a N<sup>4</sup>LO correction in the combined expansion in powers

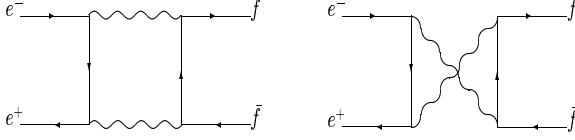


Figure 1. Direct and crossed box diagrams for  $e^+e^- \rightarrow f\bar{f}$ .

of  $\alpha$  and  $v$ , far beyond the scope of this analysis. In addition, contributions to the total cross section from diagrams with real photons emitted from the produced taus can be shown to begin at N<sup>3</sup>LO, and factorization remains at NNLO. The total cross section can thus be written as an integration over the product of separate pieces including initial, intermediate and final state corrections:

$$\sigma(s) = \int^s F(s, w) \left| \frac{1}{1 + e^2 \Pi_{\text{em}}(w)} \right|^2 \tilde{\sigma}(w) dw. \quad (2)$$

The radiation function  $F(s, w)$  [11] describes initial state radiation, including virtual corrections. The integration accounts for the effective energy loss due to photon emission from the  $e^+e^-$  pair, and it includes the largest corrections coming from the emission of an arbitrary number of initial photons, which can sizably suppress the total cross section.  $\tilde{\sigma}$  collects only final-state interactions between the tau leptons, and it is usually written in terms of the tau spectral density  $R_\tau$ ,

$$\tilde{\sigma}(e^+e^- \rightarrow \gamma^* \rightarrow \tau^+\tau^-) = R_\tau(s) \sigma_{pt}, \quad (3)$$

with  $\sigma_{pt} = \frac{4\pi\alpha^2}{3s}$ . The threshold behaviour of the total cross section will be ruled by the expansion of  $R_\tau$  at low velocities.

### 3. Non-Relativistic Corrections: NRQED

The aim of the effective field theory approach to threshold particle production is to achieve a given accuracy in the combined expansion in powers of  $\alpha$  and the velocity  $v$ . Such double expansion is needed at threshold due to the appearance of  $(\alpha/v)^n$  terms in the QED perturbative expansion at the  $n$ -th loop order, which forces one to treat  $\alpha$  and  $v$  on the same footing. A non-perturbative

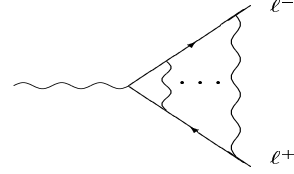


Figure 2. Ladder exchange of photons between produced fermions.

procedure to deal with such singular terms in the limit  $v \rightarrow 0$  is therefore mandatory.

The leading divergences (i.e.  $(\frac{\alpha}{v})^n$ ,  $n > 1$ ), emerging from the ladder diagrams with Coulomb photons shown in Fig. 2, are resummed in the well-known Sommerfeld factor [12]

$$|\Psi_{c,E}(0)|^2 = \frac{\alpha\pi/v}{1 - \exp(-\alpha\pi/v)}, \quad (4)$$

multiplying the Born cross section (1). Next-to-leading order (NLO) entails terms proportional to  $(\alpha/v)^n \times [\alpha, v]$ , while NNLO accuracy stands for contributions  $(\alpha/v)^n \times [\alpha^2, v^2, \alpha v]$ .

A systematic way to calculate these higher-order corrections in this regime requires the use of a simplified theory which keeps the relevant physics at the scale  $Mv \sim M\alpha$ , characteristic of the Coulomb interaction. NRQED [6] was designed precisely for this purpose. It is an effective field theory of QED at low energies, applicable to fermions in non-relativistic regimes, i.e. with typical momenta  $p/M \sim v \ll 1$ . Interactions contained in the NRQED Lagrangian have a definite velocity counting but propagators and loop integrations can also generate powers of  $v$ . With appropriate counting rules at hand, one can prove that all interactions between the non-relativistic pair  $\tau^+\tau^-$  can be described up to NNLO in terms of time-independent potentials [13], derived from the low-energy Lagrangian. Therefore, the low-energy expression of the  $\tau$  spectral density is related with the non-relativistic Green's functions [14]:

$$R_\tau^{\text{NNLO}}(s) = \frac{6\pi}{M^2} \text{Im} \left( C_1 G(E) - \frac{4E}{3M} G_c(E) \right) \quad (5)$$

with  $C_1$  a short distance coefficient to be determined by matching full and effective theory re-

sults and  $E = m_\tau v^2$  the non-relativistic energy. The details of this derivation can be found in the Appendix B of [9]. The Green's function  $G$  obeys the Schrödinger equation corresponding to a two-body system interacting through potentials derived from  $\mathcal{L}_{\text{NRQED}}$  at NNLO:

$$\left( -\frac{\nabla^2}{M} - \frac{\nabla^4}{4M^3} + V_c(\mathbf{r}) + V_{\text{BF}}(\mathbf{r}) + V_{\text{An}}(\mathbf{r}) - E \right) \times G(\mathbf{r}, \mathbf{r}', E) = \delta^{(3)}(\mathbf{r} - \mathbf{r}') \quad (6)$$

The term  $-\frac{\nabla^4}{4M^3}$  is the first relativistic correction to the kinetic energy.  $V_c$  stands for the Coulomb potential with  $\mathcal{O}(\alpha^2)$  corrections. At NNLO, the heavy leptons are only produced in triplet S-wave states, so we just need to consider the corresponding projection of the Breit-Fermi potential  $V_{\text{BF}}$ . Finally,  $V_{\text{An}}$  is a NNLO piece derived from a contact term in  $\mathcal{L}_{\text{NRQED}}$ , which reproduces the QED tree level s-channel diagram for the process  $\tau^+\tau^- \rightarrow \tau^+\tau^-$ .

A solution of eq. (6) must rely on numerical or perturbative techniques. In the QED case, a significant difference between both approaches is not expected, being  $\alpha$  such a small parameter. Consequently we follow the perturbative approach, using recent results [14], where NLO and NNLO corrections to the Green's function are calculated analytically, via Rayleigh-Schrödinger time-independent perturbation theory around the known LO Coulomb Green's function  $G_c$ . We refer the reader to Appendix C of [9] for complete expressions of the Green's function corrections.

#### 4. Vacuum Polarization

For a complete NNLO description of  $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$ , two-loop corrections to the photon propagator should be included. The light lepton contributions to the vacuum polarization are the standard 1- and 2-loop perturbative expressions. For the  $\tau$  contribution in the threshold vicinity  $q^2 \gtrsim 4M^2$ , resummation of singular terms in the limit  $v \rightarrow 0$  is again mandatory. Under the assumption  $\alpha \sim v$ , we need to know NLO contributions to  $\Pi_\tau(q^2)$ , performing the direct matching for both real and imaginary parts.

In the hadronic sector, we can relate the hadronic vacuum polarization with the total cross

section  $\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \text{had})$ . Below 1 GeV, the electromagnetic production of hadrons is dominated by the  $\rho$  resonance and its decay to two charged pions. The photon mediated  $\pi^+\pi^-$  production cross section is driven by the pion electromagnetic form factor  $F(s)$ . An analytic expression for  $F(s)$  was obtained in Ref. [15], using Resonance Chiral Theory and the restrictions imposed by analyticity and unitarity. The obtained  $F(s)$  provides an excellent description of experimental data up to energies of the order  $s_\rho \sim 1 \text{ GeV}^2$ .

For the integration region above  $s_\rho$ , we use  $\text{Im}\Pi_{\text{had}}$  as calculated from pQCD. Our simple estimate has been proved [9] to deviate by less than 5% for the running of  $\alpha$  at the scale  $\sqrt{s} = M_Z$ . Considering that  $\Pi_{\text{had}}$  modifies  $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$  near threshold by roughly 1%, our result has a global uncertainty smaller than 0.1% for the total cross section. Clearly, our ansatz could be easily improved using a more realistic hadronic spectrum, and, indeed, most experiments have their own routines to accurately implement vacuum polarization together with initial state radiation in their data analyses.

#### 5. Electroweak and Bound States effect

Electroweak production of heavy quarks including threshold effects has already been studied in previous papers [16], and can be easily incorporated in our basic formula (2). However, the characteristic  $m_\tau^2/M_Z^2$  suppression of this set of corrections and the small value of the neutral-current couplings of the leptons make these terms fully negligible in our analysis.

Green's functions develop energy poles below threshold corresponding to spin triplet ( $n^3S_1$ ) electromagnetic  $\tau^+\tau^-$  bound states. The small width of these bound levels, dominated by their  $e^+e^-$  decay rate

$$\Gamma_{ee} = \frac{m_\tau \alpha^5}{6n^3} \approx \frac{6.1 \cdot 10^{-3} \text{ eV}}{n^3}, \quad (7)$$

make these states very difficult to be resolved experimentally. For the same reason, the bound states cannot affect the shape of the cross section at threshold, contrary to the case of heavy quark

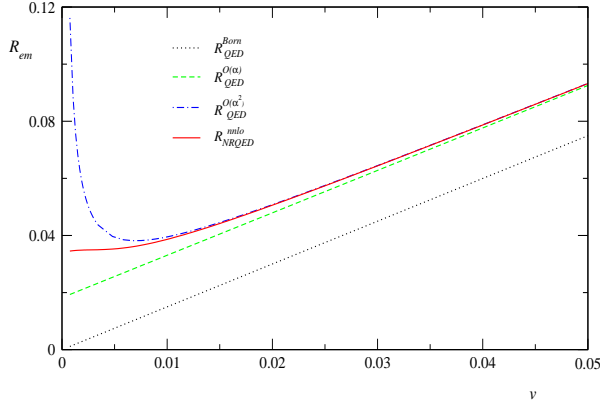


Figure 3. The spectral density  $R_\tau$  at low velocities in both QED and NRQED.

threshold production, where bound states play a crucial role.

## 6. Numerical analysis for $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$

The need for performing resummations of the leading non-relativistic terms  $(\alpha/v)^n [v, v\alpha, v^2, \dots]$  is evidenced in Fig. 3. The QED spectral density vanishes as  $v \rightarrow 0$ , due to the phase space velocity in formula (1), which is canceled by the first  $v^{-1}$  term appearing in the  $\mathcal{O}(\alpha)$  correction, making the cross section at threshold finite. More singular terms near threshold,  $v^{-2}, \dots$  arising in higher-order corrections completely spoil the expected good convergence of the QED perturbative series at low  $v$ . This is no longer the case for the effective theory perturbative series, whose convergence improves as we approach the threshold point, as shown in Fig. 4, and higher-order corrections reduce the perturbative uncertainty. In the whole energy range displayed in Fig. 4, the differences between the NNLO, NLO and LO results are below 0.8%, which indicates that the LO result, i.e. the Sommerfeld factor, contains the relevant physics to describe the threshold region. Adding the intermediate and initial state corrections we have a complete description of the total cross section of  $\tau^+\tau^-$  production, as shown in Fig. 5. Coulomb interaction between the produced  $\tau$ 's, becomes essential within few MeV's above the threshold, and its effects have to be taken into account to

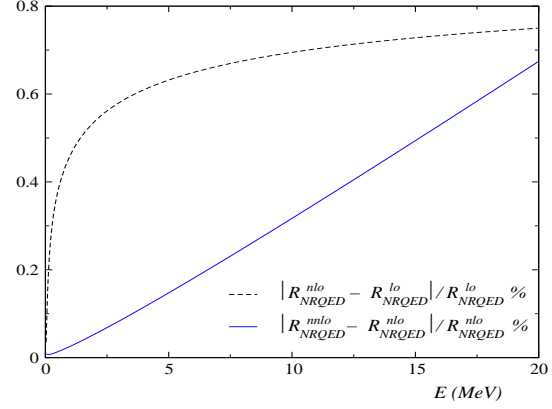


Figure 4. Relative sizes of corrections to  $R_\tau(s)$  as calculated in NRQED.

all orders. Initial state radiation effectively reduces the available center-of-mass energy for  $\tau$  production, lowering in this way the total cross section.

We should emphasize that NNLO corrections do not modify the predicted behaviour of the LO and NLO cross section, but are essential to guarantee that the truncated perturbative series at NLO gets small corrections from higher-order terms. Differences with previous evaluations mainly concern the correct resummation of NNLO terms  $\propto (\alpha/v)^n \times [\alpha^2, v^2, \alpha v]$ , which on perturbative grounds means that our results should differ by  $\alpha^2$  terms. The latter are seemingly negligible, but one has to check that all-order resummation of them does not produce any unexpected enhancement.

In addition, factorized formulas proposed in previous works do not have under control the NNLO terms being resummed beyond the order in  $\alpha$  at which the matching between the non-relativistic and the QED results is performed. These fake resummations may be harmless in QED, being  $\alpha$  such a small parameter, but must be handled with care when estimating the uncertainties of the theoretical calculation. In that sense, notice in Fig. 4 that NNLO corrections to the spectral density are not  $\mathcal{O}(\alpha^2) \sim 0.005\%$  with respect to the leading-order, as we would naively estimate. Hence, we can safely conclude that the theoretical uncertainty of our analysis of  $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$  at energies close to threshold

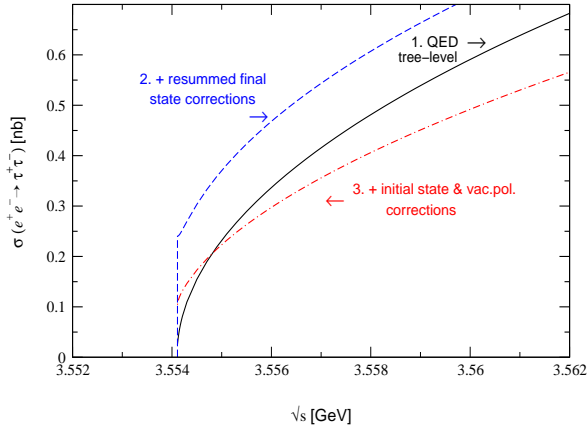


Figure 5. Total cross section  $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$  at threshold.

is lower than 0.1%.

A FORTRAN code which evaluates the spectral density  $R_\tau^{\text{NNLO}}(s)$  will be soon available at <http://alpha.ific.uv.es/~ruiz>

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