Dark Energy from Strings

- Paul H. Frampton
  (UNC-Chapel Hill)

- hep-th/0110167
  +M. Bastero-Gil + L. Mersini

Modified Dispersion Relations from Closed Strings in Toroidal Cosmology

- Paul H. Frampton
  (UNC-Chapel Hill)

- WIN 2002
- University of Canterbury
  Christchurch
- January 24, 2002

- hep-th/0110167
  +M. Bastero-Gil + L. Mersini

CONTENTS

INTRODUCTION

- A quantitative argument about the origin of dark energy.
- Blue shift -> transPlackian energies.
- Freeze-out of ultra-low frequencies.
- Key role played by winding modes of closed strings.

Outline

- First we discuss FRW cosmological solutions for string theory in a D-dimensional torus.
- Then the quantum hamiltonian from closed string theory uses a correspondence principle between string and quantum operators.
- Coarse-graining is used to describe the evolution of the system in an expanding universe.
- The end-result is a dispersion relation between frequency and wave number decreasing exponentially for very large wave number.

Coarse graining

- Choose 3 expanding dimensions and D-3 compactified.
- String scale is UV lattice cutoff.
- Divide into system and environment.
- Integrate out environmental degrees of freedom.
- Non-equilibrium dynamics described by such sequential rescaling as universe expands.

2. TOROIDAL STRING COSMOLOGY

- An interesting string cosmological scenario was proposed by Brandenberger and Vafa in 1988.
- We follow the spirit of BV, but look more closely at the winding-momentum correlation for closed strings.
- One may use T duality of closed strings to argue that there is a maximum temperature and before that one uses the T-dual description with R -> 1/R.
- T-duality is an exact symmetry under R <->1/R and m <-> w.

POSSIBLE REASON FOR D=3

- As the D uncompactified dimensions expand the winding modes must decrease in number by annihilation.
- Strings do not generally meet in $D > 3$ so the necessary annihilation cannot take place and the expansion will stop.
- To avoid a quick stop to the expansion one needs $D = 3$ (or less).
- We will phenomenologically choose the maximum value D=3 in our toroidal solution.

Cosmological solutions on T^D were found by M. Mueller in 1989. He studied the cosmology of bosonic strings propagating in the background field defined by a time-dependent dilaton field $\Phi(t)$ and spacetime metric:

$$ds_d^2 = G_{\mu\nu}(X)dX^\mu dX^\nu = -dt^2 + \sum_{i=1}^{D} 4\pi R_i^2(t) dX_i^2$$

The radii $R_i(t)$ are scale-dependent scale factors to be used later in coarse-graining.

The equations of motion of the bosonic string in background fields are obtained from the action:

$$I = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g}$$

$$\left[g^{mn} G_{\mu\nu}(X) \partial_m X^\mu \partial_n X^\nu + \frac{1}{2}\alpha' \Phi R^{(2)}\right]$$

Freedom from Weyl anomalies leads to the background field equations:

$$\beta^G_{\mu\nu} = R_{\mu\nu} + \nabla_\mu \nabla_\nu \Phi = 0$$

$$\beta^\Phi = \frac{d-26}{3\alpha'} - R + (\nabla\Phi)^2 - 2\nabla^2\Phi = 0$$

Using the toroidal metric these reduce to:

$$\ddot{\Phi} - \Sigma \frac{\ddot{R}_i}{R_i} = 0$$

$$\frac{\ddot{R}_i}{R_i} + \Sigma_{j \neq i} \frac{\dot{R}_i \dot{R}_j}{R_i R_j} - \Phi \frac{\dot{R}_i}{R_i} = 0$$

$$\ddot{\Phi} - \frac{1}{2}\dot{\Phi}^2 + \Sigma_{i<j} \frac{\dot{R}_i \dot{R}_j}{R_i R_j} = \frac{d-26}{3\alpha'}$$

The solutions are of the form

$$e^{-\Phi(t)} \sim t^p \qquad R_i(t) \sim t^{p_i}$$

with constraints

$$\Sigma_{i=1}^{D} p_i^2 = 1 \qquad \Sigma_{i=1}^{D} p_i = 1 - p$$

From this complex of solutions, we can choose D=3 expanding (common p_i) and D-3 compactified (e.g. common p_j) dimensions.

Because of the toroidal topology the three expanding dimensions contain both types of modes: momentum and winding. To understand the dynamics we need to discuss the formulation of a quantum hamiltonian describing such (noninteracting) strings on this background. Such a hamiltonian framework has been developed by mathematical physicists already in the nineties.

3. QUANTUM HAMILTONIAN

- For classical strings exact solutions are known for a variety of backgrounds in closed string field theory.
- They are found by solution of a two-dimensional non-linear sigma model.
- In particular, for the present case a quantum hamiltonian whose KE terms give the zero modes and interactions generate string excitations and correlations.
- This describes first-quantized weakly coupled strings.

We include the dynamics of both modes, momentum modes $p_{1,i} = m/\bar{R}_i$ and winding modes with momenta $p_{2,i} = w/\bar{R}_i$.

Here we have defined the dimensionless quantity $\bar{R}_i = R_i(\alpha')^{-1/2}$.

We *choose* a cosmology with three toroidal radii equal and large $\bar{R}_i \gg 1$ in Planckian units (or string scale units) and the other $(D-3)$ toroidal radii equal and small.

Thus $R(t) = a(t)$ becomes the scale factor in the $3+1$ metric while R_c corresponds to the radius in this factorizable metric of the $(D-3)$ compact dimensions z_j :

$$\begin{aligned} ds_D^2 &= -dt^2 + 4\pi R^2(t)dx_i^2 + 4\pi R_C^2(t)dz_j^2 \\ &= a(\eta)[-d\eta^2 + dy^2] + ds_{D-3}^2 \end{aligned}$$

The toroidal solutions of Mueller give the time dependences of the radii:

$$R(t) = \alpha_U t^{p_U}$$

$$R_C(t) = \alpha_C t^{p_C}$$

The values of p_U and p_C depend on the dimensionality D in an interesting way. If we assume, for example, that the dilaton is time-independent ($p = 0$) and that the compactification is isotropic we find that for $4 \leq D < \infty$ then $0.5 \leq p_U < 1/\sqrt{3} \simeq 0.577$.

If $D = 4$ the scale fator behaves as in a radiation-dominated universe. If $D \geq 5$ we can assume that $(D-4)$ additional dimensions have $p_C^{'} \ll p_C$ to achieve the same result. In what follows we will not need to specialize to this particular solution.

For the dark energy, we have in mind the correlation between momentum and winding modes. The kinetic energy terms are well-known but how best to characterize the interactions between these two types of modes? Some important aspects of the problem were addressed and partially solved by: K.Kikkawa *et al* Prog. Theor. Phys. **98,** 687 (1997).

At temperatures T much below the string T_{Planck} there is little correlation but for $T \sim T_{Planck}$ there is.

The partition function can be calculated by:

$$Z = \Sigma_\sigma \exp(-n_\sigma \epsilon_\sigma)$$

where n_σ is the number of strings in state σ with energy ϵ_σ with:

$$\epsilon_\sigma = p_0 = \sqrt{\left(\frac{m}{R}\right)^2 + ((wR)^2 + N + \tilde{N} - 2}$$

Here σ counts over (m, w) with the crucial constraint $N - \tilde{N} = mw$. N and $\tilde{N}$ are sums over left- and right- moving string excitations.

By now, we are considering only 3 large spatial dimensions.

For the string state in place of p_1 and p_2 for the momentum and winding respectively it is advantageous to use fields with momenta $k_L = p_1 + p_2$ and $k_R = p_1 - p_2$ as can be seen from the exact solutions of *e.g.* Tseytlin [A.A. Tseytlin, Class. Quant. Grav. **12**, 2365 (1995)].

In writing a CGEA (Coarse-Grained Effective Action) the kinetic terms are unambiguous and the interaction terms in the exact classical solution respect surprising simplicity (quartic only) and T-duality, the latter being extremely restrictive.

Explicitly the quantum hamiltonian for a particular example with uncompactified x_1 and x_2 written in polar coordinates $x_1 + ix_2 = \rho e^{i\phi}$ and x_3 also uncompactified (but could be compactified along with similar coordinates) together with time and one additional compactified dimension $y \subset (0, 2\pi R)$.

Here $J_{L,R}$ are bilinear quadratic operators. J_R^2 is quartic.

17.

$$
\begin{aligned}
\hat{H} &= \hat{L}_0 + \hat{\tilde{L}}_0 \\
&= \frac{1}{2}\alpha' \left(-E^2 + p_\alpha^2 + \frac{1}{2}(Q_+^2 + Q_-^2) \right) \\
&\quad + N + \tilde{N} - 2c_0 \\
&\quad - \alpha' [(q + \beta)Q_+ + \beta E] - \\
&\quad - \alpha' [(q - \alpha)Q_- + \alpha E] J_L \\
&\quad \frac{1}{2}\alpha' q \left[(q + 2\beta) J_R^2 + \right. \\
&\quad + (q - 2\alpha) J_L^2 + \\
&\quad \left. 2(q + \beta - \alpha) J_R J_L \right]
\end{aligned}
$$

For an exact solution for the hamiltonian of the string in a toroidal background a quartic potential energy was advocated and found in

J.G. Russo and A.A. Tseytlin, Nucl. Phys. **B448**, 293 (1995).

We accept this indication that for closed strings on a torus the quantum hamiltonian contains only quartic interactions for our present case of $(T_3) \times (T_{(D-3)}) \times (\text{time})$.

Such a hamiltonian is, however, for a static background, *i.e.* a constant scale factor $R(t)$.

So we now explain our T-dual calculation in the coarse-grained effective action (CGEA) formalism where the dynamics of an expanding background is replaced by scaling on a static background.

4. COARSE-GRAINING

- The dynamics is non-equilibrium due to the expanding background spacetime.
- All information about the evolution of the momentum and winding modes will be contained in the effective action.
- The path integral will be written in terms of the quantum fields described above.
- We work in a conformally flat background metric as characterized earlier.
- Rescaling of the conformal factor (a) will be accomplished through the coarse-graining technique.

The conformal background is:

$$ds_D^2 = a(\eta)^2[-d\eta^2 + dy^2] + ds_{d-3}^2$$

We define a momentum field $\phi_1(R,x)$ and a winding field $\phi_2(R,x)$ such that:

$$\phi_i(x) = \int e^{ip_i x}\phi_i(p_i)d^3p_i$$

$$\int |\nabla\phi_i|^2 d^3x = \int d^3p_i p_i^2 \phi_i(p_i)$$

where

$$\nabla = R\partial/\partial x = \partial/\partial y$$

and $p_i = p_1, p_2$.

Define also:

$$\psi_{L,R}(R,x) = \phi_1(R,x) \pm \phi_2(R,x)$$

Similarly there is another set of fields $\psi_{C,a}$ that are functions of the compactified dimensions z_a.

However, their energy contribution to the total hamiltonian density is proportional to the volume of the compact space and to p_a^2 where p_a are the components of momenta in the additional dimensions. This merely adds a constant to the logarithm of the partition function.

Thus we focus on the fields $\psi_{L,R}$ living in our 3+1 spacetime.

The quantum hamiltonian that describes the energy of our two string modes in the D=3 expanding dimensions including both the KE piece $\mathcal{H} = L_0 + \tilde{L}_0$ and the higher string excitations $(N + \tilde{N} - 2)/(\alpha')$ is similar to that for spin waves on a dual lattice.

It can be written:

$$\begin{aligned}\mathcal{H}_3 = &|\nabla\psi_L|^2 + |\nabla\psi_R|^2 \\ &+m_0^2(|\psi_L|^2 + |\psi_R|^2) \\ &+g_1(|\psi_L|^4 + |\psi_R|^4) + g_2|\psi_L|^2|\psi_R|^2\end{aligned} \tag{1}$$

where

$$\psi_L = \Sigma(u_n b_n + u_n^* b_n^\dagger), \quad \psi_R = \Sigma(\tilde{u}_n \tilde{b}_n + \tilde{U}_n^* \tilde{b}_n^\dagger)$$

The periodic lattice condition

$$N - \tilde{N} = \Sigma m_i w_i$$

introduces an interaction term of the form $\nabla\psi_L \nabla\psi_R$. In terms of the two-component state $|\Psi >= |\psi_1, \psi_2 >$ the hamiltonian reads:

$$\begin{aligned} \mathcal{H} = & |\nabla\Psi|^2 + \nabla\psi \hat{X} \nabla\Psi + m_0^2 |\Psi|^2 \\ & + g_1 |\Psi|^4 + (g_2 - 2g_1) |\Psi \hat{X} \Psi|^2 \end{aligned} \tag{1}$$

where

$$\hat{X} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

This system is known as the dual momentum-space lattice and for $g_2 = 2g_1$ reduces to the XYZ-model of condensed matter theory. We shall restrict to the XYZ model for simplicity ($g_2 = 2g_1$) for the rest of the talk.

Such a periodic lattice system has a solution with respect to lattice translation invariance $\sim e^{-pl}$ with the lattice spacing equal to the string scale $\sqrt{\alpha'}$.

The interaction term lifts the degeneracy between energy eigenstates and as a result the gap energy produced between the ground state and higher excitation states is:

$$p^2\Delta_p = p^2|\cos(2\theta)| = p^2|2\cos^2(\theta) - 1|$$

in which

$$pl = p\sqrt{\alpha'} = \sqrt{\alpha'(p_1^2 + p_2^2)} = \sqrt{\left(\frac{m}{\bar{R}}\right)^2 + (w\bar{R})^2}$$

and $\theta \to \theta + ipl$. Therefore

$$z_p = 1 + \Delta_p \leq 2\cosh^2(pl)$$

The gap energy introduces a correction to the kinetic energy such that in momentum space the hamiltonian reads

$$\mathcal{H} = z_p p^2 |\Psi|^2 + m_0^2 |\Psi|^2 + g_1 |\Psi|^4$$

with $z_p = 1 + \Delta_p$.

In order to recover the canonically normalized kinetic term we renormalize:

$$\Psi \to \tilde{\Psi} = z_p^{1/2} \Psi$$

$$m_0^2 \to \tilde{m}_0^2 = z_p^{-1} m_0^2 = \frac{m_0^2}{1 + \Delta_p}$$

$$g_1 \to \tilde{g}_1 = z_p^{-2} g_1$$

whereupon:

$$\mathcal{H} = |\nabla \tilde{\Psi}|^2 + \tilde{m}_0^2 |\tilde{\Psi}|^2 + \tilde{g}_1 |\tilde{\Psi}|^4$$

The action in all D spatial dimensions is therefore:

$$
\begin{aligned}
S_D = & \int dt R_C d^{(D-3)} z_a a(t) d^3x (\mathcal{H}_3[\tilde{\Psi}(x)] + \\
& + \mathcal{H}_C[\Psi_{C,a}(z)]) \\
= & V_C \int a(t) dt d^3x \mathcal{H}_3[\tilde{\Psi}(x)] + \\
& + V_U \int R_C dt d^{(D-3)} z \mathcal{H}_C[\Psi_{C,a}(z)]
\end{aligned}
$$

where V_U, V_C are the volumes obtained by integrating out the compact, uncompactified dimensions respectively.

The partition function is:

$$Z = Z_C Z_3$$

$$Z_C = \int D\Psi_{C,a} e^{-V_U \int R_C dt d^{(D-3)} z \mathcal{H}_C[\Psi_{C,a}]}$$

$$Z_3 = \int D\tilde{\Psi} e^{-V_C \int a(t) dt d^3x \mathcal{H}_3[\tilde{\Psi}]}$$

The contribution of Z_C is a simple Gaussian with the result:

$$\begin{aligned} Z_C &= \int D\Psi_{C,a} e^{-V_U \int d^{(D-3)} p_a A_a \Psi_{C,a} p_a^2 \Psi_{C,a}} \\ &= \Pi_a \sqrt{\frac{\pi}{A_a V_U}} \end{aligned}$$

leading to

$$Z = N' Z_3$$

with the volume V_C reabsorbed into the parameters of $\mathcal{H}_3$ in Z_3.

Non-equilibrium dynamics

- Want a simplified description for the dynamics of our non-equilibrium system consisting of both momentum and winding modes.
- This is done by carrying out the necessary steps of "coarse-graining":
- 1) distinguish the system from the environment.
- 2)coarse-grain the environment.
- 3) measure the influence of (2) on the system giving an effective dynamics.

COARSE-GRAINING FOR D=3.

We separate our modes into system (S) and environment (E) and integrate out the degrees of freedom for E.

The environment (E) is chosen as all short wavelength modes with momenta:

$$\frac{\Lambda}{b} < p^E = \frac{1}{\sqrt{\alpha'}}[(m/R)^2 + (wR)^2]^{1/2} < \Lambda$$

$b = a(t)/a(0)$ and $a(t)$ plays the role of the collective coordinate describing the E degrees of freedom. The non-equilibrium dynamics is replaced by this scaling. The system (S) modes satify: $p^S < \frac{\Lambda}{b}$. At initial times when $b = 1$ all our modes are in the system. As time evolves, $b > 1$ and modes systematically transfer from S to E. As t becomes large all the modes except $m \leq R\Lambda, w = 0$ have transferred to E.

After splitting the modes into S + E the action can be separated as:

$$S[\tilde{\Psi}] = S_S[\tilde{\Psi}_S] + S_0[\Psi_E] + S_I[\tilde{\Psi}_E, \tilde{\Psi}_S]$$

where

$$S_S = \int a(t)dt \int d^3x(\Psi_S G_S^{-1}\tilde{\Psi}_S + g_1\tilde{\Psi}_S^4)$$

$$S_0 = \int a(t)dt \int d^3x\tilde{\Psi}_E G_E^{-1}\tilde{\Psi}_E$$

$$S_I = \int a(t)dt \int d^3x g_1 \left[4\tilde{\Psi}_S^3\tilde{\Psi}_E + 6\tilde{\Psi}_S^2\tilde{\Psi}_E^2 + 4\tilde{\Psi}_S\tilde{\Psi}_E^3 + \tilde{\Psi}_E^4\right]$$

The S and E Green's functions are respectively:

$$G_S[p^S < \Lambda/b] = [(p^S)^2 + \tilde{m}_0^2]^{-1}$$

$$G_E[p^E > \Lambda/b] = [(p^E)^2 + \tilde{m}_0^2]^{-1}$$

The total Green's function is thus:

$$G[p] = G_S + G_E$$

After integrating out the high energy modes we are left with an effective action S_{eff} which depnds only on the system variables $p^S < \Lambda/b$ such that:

$$S_{eff}[\tilde{\Psi}_S] = S_S[\tilde{\Psi}_S] + \Delta S[\tilde{\Psi}_S]$$

The term ΔS results from the interaction of the system with the environment.

The ΔS modifies the parameters in S_{eff} by:

$$\tilde{m}^2 = \tilde{m}_0^2 + \delta \tilde{m}_0^2 \qquad \tilde{g} = \tilde{g}_1 + \delta \tilde{g}_1$$

(Some more details of this are in the Appendix of the BFM paper or in the papers of Bei-Lok Hu)

We rescale variables in S_{eff} by:

$$p^{'} = bp \qquad \tilde{\Psi}^{'}(p^{'}) = b^{-(D+2)/2}\tilde{\Psi}(p^{'}/b)$$

The original cutoff Λ and range of momenta are restored after resclaing of parameters in a static spacetime. The procedure is repeated n times for small time increments between the initial and final times.

$$S_{eff}(\tilde{\Psi}') = b^{-D} \int d^D p' \tilde{\Psi}(p'/b) \left[\left(\frac{p'}{b} \right)^2 + \right.$$

$$\left. + \tilde{m}^2 + \tilde{g} < \tilde{\Psi}^2 > \right] \tilde{\Psi}(p'/b)$$

$$= \int d^D p' \tilde{\Psi}(p') \left[(p')^2 + \right.$$

$$\left. b^2 \tilde{m}^2 + b^{4-D} \tilde{g} < \tilde{\Psi}^2 > \right] \tilde{\Psi}'(p')$$

To have the same functional form after rescaling requires that we redefine:

$$\tilde{m}'^2 = b^2 \tilde{m}^2 \qquad \tilde{g}' = b^{4-D} \tilde{g}$$

Repeating this procedure n times and then letting $n \to \infty$ results in RG equations for couplings.

The dispersion relation for the fluctuations is proportional to the inverse of the two-point function. The two-point function shows that the dispersed frequency $G^{-1}[p] = \omega[p]$ has the ultraviolet behavior

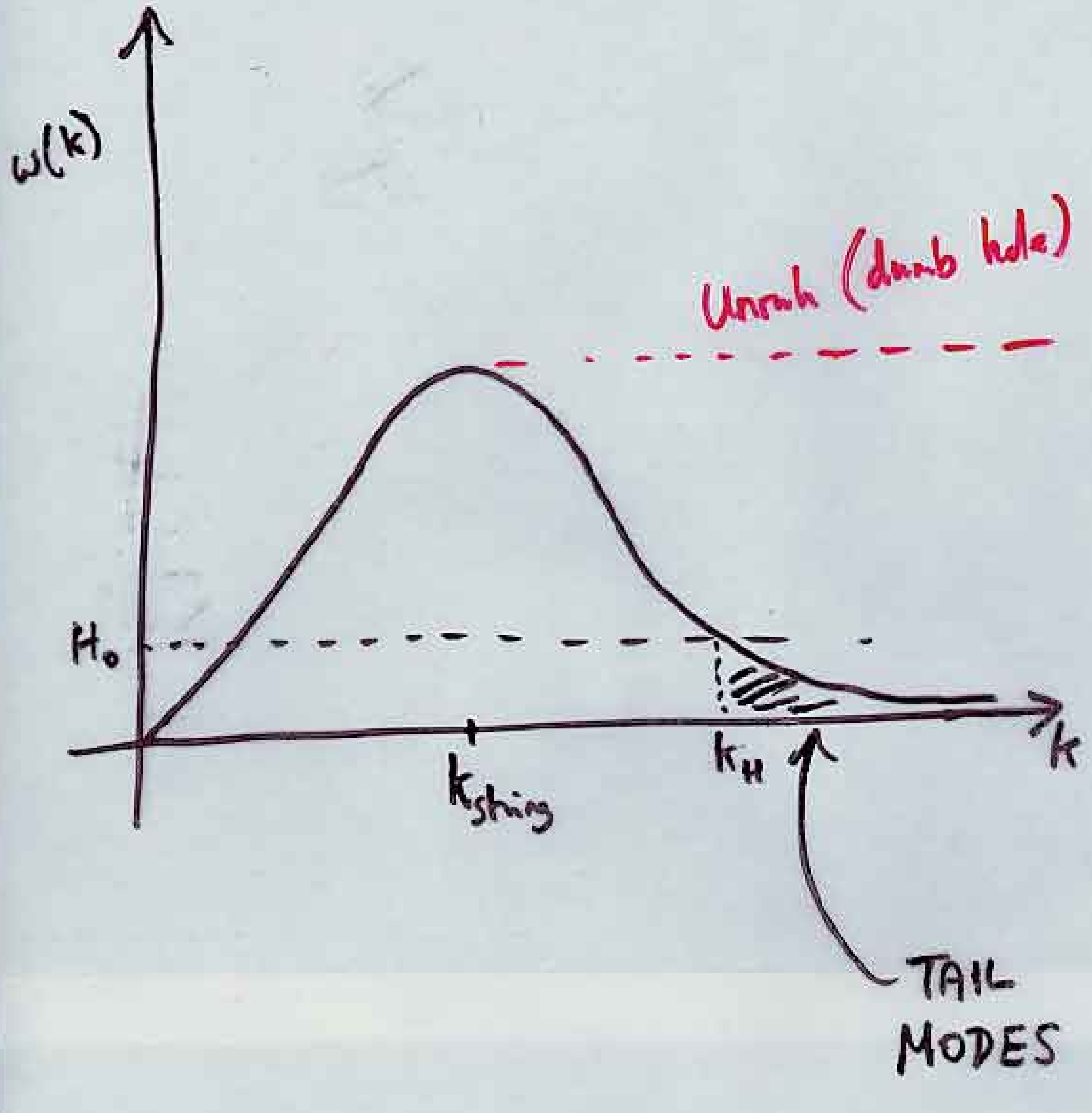
$$\omega[p] \overset{p \to \infty}{\rightarrow} \frac{m_0^2}{2\cosh^2 p\sqrt{\alpha'}} \simeq e^{-2\sqrt{\alpha'}p}$$

We may understand this physically by observing that just as correlation functions fall off exponentially in x-space, here on the dual lattice they fall exponentially in p-space. This is related to T-duality of the string theory.

The above result leads to our interpretation of the dark energy.

5. DARK ENERGY

- We have seen that the correlation between momentum modes and winding modes leads to a dispersion relation which fall exponentially at high k.
- Now we assert out interpretation of the dark energy as frozen-in non-dynamical modes.
- Those modes with frequency below the present Hubble constant are relevant.


ω(k)
Unruh (dumb hole)
H_0
k_{string}
k_H
k
TAIL
MODES

From the Figure and using that the occupancy numbers give factors of order one (a separate calculation) the fraction of the total energy in tail modes is:

$$\frac{\rho_{tail-modes}}{\rho_{total}} = \frac{\int_{k_H}^{\infty} kdk \int \omega(k) d\omega}{\int_0^{\infty} kdk \int \omega(k) d\omega}$$

Focus on the decisive numerator:

$$\int kdk \frac{e^{-2\sqrt{\alpha'}k}}{-2\sqrt{\alpha'}k} = [(k_H 2\sqrt{\alpha'} + 1](H_0\sqrt{\alpha'})^2$$

Using:

$$e^{-2\sqrt{\alpha'}k_H} = H_0^2\alpha'$$

We find that, up to power law prefactors, the basic result is that

$$\frac{\rho_{tail-modes}}{\rho_{total}} = \left(\frac{H_0}{M_{Planck}}\right)^2 \simeq 10^{-120}$$

This is the main result of this work.

DARK ENERGY FROM STRINGS

- In the BV model there are both modes - momentum and winding at all stages in the toroidal evolution of the cosmology.
- At the string temperature (the maximum temperature) there is a strong correlation between them due to the BCs.
- The non-equilibrium dynamics, computed by coarse graining, implies a frozen-in uniform condensate of the winding modes with correct density for Dark Energy.

ALTERNATIVE FOR DARK ENERGY

- The cosmological constant of Einstein (1917) involves extreme fine-tuning.

- Quintessence involves a slowly-varying scalar field, is arbitrary and possesses degenerate solutions for the same cosmic parameters.

STRINGY DARK ENERGY

- Has some definite advantages:

- Gives a satisfactory and intuitive fate of the winding modes of the BV model.

- Contains no fine tuning since the answer is simply expressed in terms of two quantities, the Hubble constant and the Planck mass, both known since the 1920's.

- Preferable to cosmological constant or quintessence.