Crystalline Color Superconductivity in Dense Matter

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1 Introduction

In this talk I will discuss a recently proposed color superconducting phase of asymmetric quark matter where the up and down quark have different chemical potential, being in chemical equilibrium with electrons. Using Schwinger-Dyson equations derived from an effective theory of low-energy quasiparticles, we examine both the case of an effective four Fermi interaction (appropriate for intermediate densities, such as those found in a neutron star) and Landau-damped one gluon exchange (appropriate for asymptotic density) [1]. We then briefly discuss patching together regions of plane-wave (LOFF) condensation.

One of the most intriguing problems in QCD is to understand how matter behaves at extreme densities, densities much higher than the nuclear density, $\rho_0 = 0.16\ \mathrm{fm}^{-3}$, as expected in the core of compact stars like neutron stars or in the relativistic heavy ion collision (RHIC). According to QCD, which is now firmly believed to be the theory of strong interaction, the interaction among quarks becomes weaker and weaker as they get closer and closer. Therefore, when nucleons are closely packed, the wavefunctions of quarks inside nucleons will overlap with each other at high nucleon density and hadronic matter will turn into quark matter. The physical properties of quark matter are questions we wish to answer.

In general, weakly interacting fermion matter will be a Fermi liquid, forming a Fermi surface. However, according the Cooper theorem, the IR fixed point of the system of weakly interacting fermions will be very different from the Fermi liquid, if there exists an attraction between fermions with opposite momentum, even for an arbitrarily weak attraction. For quark matter, quarks attract not only holes but also quarks themselves if scattering occurs in the color anti-triplet channel. This can be seen by calculating the Coulomb potential due to one-gluon exchange interaction, which is valid at high density;

$$V(r) = -i \int \frac{\mathrm{d}^3\mathrm{q}}{(2\pi)^3}\, T^A D_{00}^{AB}(\vec{q}, q_0 = 0) T^B e^{i\vec{q}\cdot\vec{x}}, \tag{1}$$

where T^A's are $SU(3)_c$ generators and $D^{AB}_{\mu\nu}$ is the gluon propagator. In perturbative QCD, $D^{AB}_{00} = i\delta^{AB}/\vec{q}^2$ and we see that the sign of the Coulomb potential depends on the initial (or final) color states i, j (or k, l) of quarks:

$$T^A_{ik}T^A_{jl} = -\frac{2}{3}\delta_{[ij]}\delta_{[kl]} + \frac{1}{3}\delta_{(ij)}\delta_{(kl)}. \tag{2}$$

Of course, the Coulomb potential by one-gluon exchange interaction would not apply at intermediate density where the strong coupling constant is no longer small. However, it is quite reasonable to assume that the color exchange interation is attractive even at the intermediate density for the color anti-triplet channel since it reduces the color Coulomb energy among quarks. Since attraction occurs in the color anti-triplet channel for the diquark scattering and in the color singlet channel for the quark-hole scattering, the candidates for the condensates are therefore either diquark condensates in color anti-triplet channel or quark-hole condensates in color singlet channel, given as

$$\langle\psi_i(\vec{p})\psi_j(-\vec{p})\rangle\,\widetilde{\chi}^0_0 \quad \text{or} \quad \left\langle\overline{\psi}_i(-\vec{p})\psi_j(\vec{p})\right\rangle\widetilde{\chi}^0_0. \tag{3}$$

Note that the quark-hole condensate (or density wave) carries a momentum $2\vec{p}$. If it were translationally invariant condensate, it would involve antiquarks and thus energetically not prefered to form such a condensate. Since the diquark condensate has zero total momentum, the whole Fermi surface can contribute to the diquark scattering amplitude, while the phase space of quark-hole scattering is only a small fraction of the Fermi surface due to the momentum conservation. Indeed, it is shown that the diquark condensate is energetically more preferred to density waves, though the color exchange attration is weaker, due to the big difference in the phase space [2].

Depending on the density, the diquark condensate in quark matter takes two different forms. At the intermediate density, where the strange quark is too heavy to participate in Cooper pairing, the condensate takes

$$\left\langle\psi^a_{Li}(\vec{p})\psi^b_{Lj}(-\vec{p})\right\rangle = -\left\langle\psi^a_{Ri}(\vec{p})\psi^b_{Rj}(-\vec{p})\right\rangle = \epsilon_{ab}\epsilon^{ij3}\Delta, \tag{4}$$

where a, b are the flavor indices, running from up and down quarks. In this phase, called the two-flavor superconducting (2SC) phase, the condensate is flavor singlet but color anti-triplet. The condensate does not break any flavor symmetry except the $U(1)$ baryon number, while breaking the $SU(3)$ color gauge group down to a $SU(2)$ subgroup. Since the Cooper-paring quarks should have equal and opposite momenta, the minimal energy needed to Cooper-pair a strange quark with up or down quarks is, if we neglect the interaction energy,

$$\delta E = \sqrt{p_F^2 + m_s^2} - p_F \simeq \frac{m_s^2}{2\mu}, \tag{5}$$

where $p_F \simeq \mu$ is the Fermi momentum of light quarks, almost equal to the quark chemical potential. We therefore see that the energy we gain by pairing the strange quark with light quarks, which is called Cooper-pair gap, Δ_0, has to be bigger than the minimal energy δE we have to provide for strange quarks to participate in Cooper-pairing.

On the other hand, at high density, where $\mu > m_s^2/(2\Delta_0)$, it is energetically prefered for the strange quarks to form Cooper pairs with light quarks. At such high density, it is shown that the condensate takes a so-called color-flavor locking (CFL) form [3, 4, 5]

$$\left\langle \psi^a_{Li}(\vec{p})\psi^b_{Lj}(-\vec{p})\right\rangle = -\left\langle \psi^a_{Ri}(\vec{p})\psi^b_{Rj}(-\vec{p})\right\rangle = k_1\delta^a_i\delta^b_j + k_2\delta^a_j\delta^b_i,$$

which breaks not only the color symmetry but also the chiral symmetry, leaving only the diagonal subgroup $SU(3)_V$ unbroken.

A natural place to look for the signature of color superconductivity is a dense object like compact stars. However, since the stars are electrically neutral, the chemical potentials of u, d, s quarks are not equal if electrons are present. One therefore must take into account the effect of flavor asymmetry in Cooper pairing among different flavors. At weak interaction equilibrium, $u + e^- \leftrightarrow d\,(s) + \nu$, the up and down quarks have different chemical potentials, $\mu_d - \mu_u (\equiv 2\delta\mu) = \mu_e$, if the electron chemical potential is nonzero, $\mu_e\widetilde{\chi}^0_0$. When the Fermi surface mismatch becomes large enough, the input energy to make the momentum of pairing quarks equal and opposite exceeds the Cooper gap and the BCS pairing breaks down. The critical chemical potential, at which the BCS pairing breaks, is shown to be $\delta\mu = \Delta_0/\sqrt{2}$ [6]. However, as shown by Larkin and Ovchinnikov [6], and also by Fulde and Ferrell [7], even at $\delta\mu > \Delta_0/\sqrt{2}$, diquark condensate is possible, if we allow the diquark pair to carry a momentum [8], $2\vec{q} = \vec{p}_u + \vec{p}_d$, $\left\langle \psi^i_u(\vec{p}_u)\psi^j_d(\vec{p}_d)\right\rangle = \epsilon^{ij3}\Delta(\vec{q})$.

For such pairing, the (effective) four-Fermi interaction is not marginal and thus does not lead to Landau pole or dynamical mass unless the interaction is strong enough, which is a characteristic feature in dimensions higher than (1+1) [9]. As we will see later, LOFF pairing indeed occurs in dense QCD with light flavors when the couplings are bigger than critical values for both high and intermediate density.

2 Intermediate Density

At intermediate densities, where the effective QCD coupling is large and long-range interactions are likely to be screened, we take the Lagrangian to be

$$\mathcal{L} = \overline{\psi}\left(i\partial\!\!\!/ + \mu\gamma^0\right)\psi + \frac{G}{2}\left(\overline{\psi}\psi\right)^2. \tag{6}$$

Following the high density effective theory [10], we introduce the Fermi-velocity ($\vec{v}_F$) dependent field, defined as

$$\psi(\vec{v}_F, x) \equiv e^{-i\mu\vec{v}_F\cdot\vec{x}}\psi(x), \tag{7}$$

to rewrite the Lagrangian in terms of particles near the Fermi surface:

$$\mathcal{L} = \sum_{\vec{v}_F} \psi^\dagger(\vec{v}_F, x) iV \cdot \partial\psi(\vec{v}_F, x) + \sum_{\vec{v}_F^u, \vec{v}_F^d} \frac{G}{2}\overline{\psi}\psi(\vec{v}_F^u, x)\overline{\psi}\psi(\vec{v}_F^d, x) + \cdots \tag{8}$$

where $V^\mu = (1, \vec{v}_F)$, and the ellipse denotes other four-Fermi operators involving different Fermi velocities and higher order terms in $1/\mu$ expansion. Note that the velocity dependent field carries the residual momentum l^μ, if the quark carries momentum $p^\mu = (l_0, \mu\vec{v}_F + \vec{l})$.

Introducing auxiliary fields, $\sigma(x)$, we rewrite the interaction Lagrangian as

$$\mathcal{L}_{4F} = \sigma(\vec{q}, x)\psi(\vec{v}_F^d, x)\psi(\vec{v}_F^u, x) - \frac{1}{2G}\sigma^2, \tag{9}$$

where $2\vec{q} = \mu_u\vec{v}_F^u + \mu_d\vec{v}_F^d$ is a fixed vector. The vacuum is a stationary point of the effective action, obtained by integrating over the fermions only.

$$S_{\rm eff} = -\frac{1}{2G}\int d^4x\sigma^2 - i{\rm Tr}\ln\gamma_0 \begin{pmatrix} iV_d \cdot \partial & -\sigma(\vec{q}, x) \\ -\sigma^\dagger(\vec{q}, x) & iV_u \cdot \partial \end{pmatrix}. \tag{10}$$

At the stationary points, the auxiliary field is given as $\sigma_0(\vec{q}, x) = \left\langle\psi(\vec{v}_F^u, x)\psi(\vec{v}_F^d, x)\right\rangle \equiv \Delta(\vec{q})$. Since we are looking for a plane-wave condensate with a wave vector $2\vec{q}$, σ_0 is translationally invariant. The free energy density for the LOFF phase is then given in the Euclidean space as

$$V(\Delta) = +\frac{1}{2G}\Delta^2 - \frac{1}{2}\int \frac{d^4l}{(2\pi)^4}\ln\left[1 + \frac{\Delta^2}{(l_0 - il_u)(l_0 - il_d)}\right], \tag{11}$$

where we introduced new variables $l_u \equiv \vec{v}_F^u \cdot \vec{l}$ and $l_d \equiv \vec{v}_F^d \cdot \vec{l}$. Minimizing the free energy, we get the LOFF gap equation;

$$0 = \frac{\partial V}{\partial\Delta(\vec{q})} = \frac{\Delta(\vec{q})}{G} - \int \frac{{\rm d}^4l}{(2\pi)^4}\frac{\Delta(\vec{q})}{(l_0 - il_u)(l_0 - il_d) + \Delta^2}. \tag{12}$$

The characteristic feature of the gap equation (12) for the LOFF paring is that the quark propagator is a function of three independent momenta, l_0, $\vec{l}\cdot\vec{v}_F^u (\equiv l_u)$, and $\vec{l}\cdot\vec{v}_F^d (\equiv l_d)$, while in BCS pairing it is a function of two, l_0 and $\vec{l}\cdot\vec{v}_F$. In general, we may decompose $\vec{l}$ as $\vec{l} = \vec{l}_\perp + l_u\vec{v}_u^* + l_d\vec{v}_d^*$, where $\vec{v}_{u,d}^*$ are dual to $\vec{v}_F^{u,d}$, satisfying $\vec{v}_F^a \cdot \vec{v}_b^* = \delta_b^a$ with $a, b = u, d$. Though the magnitude of the Fermi velocity is $\vec{v}_F^a = 1$

for massless quarks, its dual has a magnitude $|\vec{v}_a^*| = (\sin\beta)^{-1}$, where β is the angle between $\vec{v}_F^d$ and $-\vec{v}_F^u$.

Since the quark propagator is independent of $\vec{l}_\perp$, it just labels the degeneracy in the LOFF pairing. The perpendicular momentum $\vec{l}_\perp$ forms a circle on the Fermi surface, whose radius is given as $\mu_d \sin\alpha_d$ $(= \mu_u \sin\alpha_u)$, where $\alpha_{d,u}$ are the angles between $\vec{q}$ and $\vec{v}_F^{d,u}$, respectively. Upon integrating over $\vec{l}_\perp$, the gap equation (12) becomes a (2+1) dimensional gap equation. This is in sharp contrast with the gap equation in the BCS pairing, which is (1+1) dimensional after integrating over the $\vec{l}_\perp$, namely over the whole Fermi surface.

Integrating over $\vec{l}_\perp$, we find the gap equation in Euclidean space to be

$$1 = \int_{l_0,l_u,l_d} \frac{2G\mu_d \sin\alpha_d (3\sin\beta)^{-1}}{(l_0 - il_d)(l_0 - il_u) + \Delta^2} \simeq \frac{2G\mu_d \sin\alpha_d}{3\sin\beta} \int_\Delta^{\Lambda'} \frac{\mathrm{d}l_0}{2\pi^3} \left[\frac{1}{2}\ln\left(\frac{l_0^2 + \Lambda^2}{l_0^2}\right)\right]^2, \quad (13)$$

where $(\sin\beta)^{-1}$ arises from the Jacobian and we introduced Λ as the cutoff for l_u, l_d and Λ' for l_0. In the high density effective theory, the expansion parameter is $|l^\mu|/\mu$. From the condition that $|l^\mu| < l^2/\overline{\mu}$, where $2\overline{\mu}^2 = \mu_u^2 + \mu_d^2$, we find the ultraviolet cutoff for $l_{u,d}$ is $\Lambda = \overline{\mu}\sin^2\beta$. We also take $\Lambda' = \Lambda$, since the main contribution to the gap comes from nearly on-shell quarks. Finally, integrating over l_0, the gap equation becomes

$$1 - \frac{G_c}{G} = \frac{1}{2}\left(\frac{\Delta}{\overline{\mu}\sin^2\beta}\right)\left[\ln\left(\frac{\Delta}{\overline{\mu}\sin^2\beta}\right)\right]^2, \quad (14)$$

where the critical coupling for the LOFF gap $G_c = 3\pi^3 \sin\beta/(4\mu_d\overline{\mu}\sin\alpha_d)$. For a given $\delta\mu$ and G, the LOFF gap exists only when $G > G_c$.

The best q, or equivalently the critical coupling for the LOFF pairing, is determined by minimizing the Free energy, obtained by integrating the gap equation (14) after doing the momentum integration,

$$V(\Delta) = \int_0^\Delta \frac{\partial V}{\partial \Delta'} d\Delta' \simeq \frac{1}{6G}\Delta^2\left(1 - \frac{G}{G_c}\right). \quad (15)$$

For $\overline{\mu} = 400$ MeV, $\delta\mu = 30$ MeV, $\Delta_0 = 40$ MeV, we have $\cos\beta = 0.82$, from which we obtain $\Delta = 0.076\overline{\mu}$ and $V = -1.9 \times 10^{-5}\ \overline{\mu}^2/G$.

3 Asymptotic Density

For the one-gluon exchange case, the calculation goes in parallel. The Schwinger-Dyson (SD) equation for the quark two-point function is given in the leading order in the hard-dense loop (HDL) approximation as

$$\Delta(\vec{q}, l) = (-ig_s)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} V_u^\mu D_{\mu\nu}(l-k) V_d^\nu \frac{T^a \Delta(\vec{q}, k) T^a}{k\cdot V_d k\cdot V_u - \Delta^2(\vec{q}, k)}, \quad (16)$$

where T^a is the color generator in the fundamental representation and $D_{\mu\nu}$ is the gluon propagator in the HDL approximation. Since the Landau-damped magnetic gluons give the dominant contribution, the SD equation becomes in Euclidean space as

$$\Delta(\vec{q}) = \frac{2}{3} g_s^2 \cot\beta \int \frac{\mathrm{d}l_0}{2\pi}\frac{\mathrm{d}l_\perp}{2\pi}\frac{\mathrm{d}l_u \mathrm{d}l_d}{(2\pi)^2}\frac{1}{\vec{l}^2 + \frac{\pi}{4}M^2|l_0|/|\vec{l}|}\cdot\frac{\Delta(\vec{q})}{l\cdot V_E^u l\cdot V_E^d + \Delta^2}, \quad (17)$$

where the factor 2/3 is due to the color factor and $V_E^\mu \equiv (1, -i\vec{v}_F)$. Since the high energy region $\Lambda_0 > |l_0| > 2\Lambda_u$ does not contribute the integration in Eq. (17) much, the l_0 integration is regulated by the UV cutoff of l_u or l_d integration. We then note that when $l_u^2/(\sin\beta)^2 > \alpha_s \overline{\mu}^2 |l_0|/|\vec{l}|$ the l_u integration converges rapidly. Hence for small coupling ($\alpha_s < 4\sin\beta$) we have a new UV cut-off for l_u that satisfies $\overline{\Lambda}_u = \overline{\mu}\alpha_s\sin\beta$. Integrating over $\vec{l}_\perp, l_u$ and l_d, we get

$$1 = \frac{8\cot\beta}{9\sqrt{3}\pi^2}\alpha_s^{2/3}\overline{\mu}^{-2/3}\int_\Delta^{\overline{\Lambda}_u}\frac{dl_0}{l_0^{1/3}}\left[\frac{1}{2}\ln\left(1+\frac{\overline{\Lambda}_u^2}{l_0^2}\right)\right]^2 = \left(\frac{\alpha_s}{\alpha_c}\right)^{4/3}\left[1-\frac{2}{9}\left(\frac{\Delta}{\overline{\Lambda}_u}\right)^{2/3}\left(\ln\frac{\Delta}{\overline{\Lambda}_u}\right)^2\right] \quad (18)$$

where the critical coupling for the LOFF pairing by one-gluon exchange interaction is $\alpha_c = \left(\pi^2/2\sqrt{3}\right)^{3/4}\left(\sin\beta/\cos^3\beta\right)^{1/4}$.

Now, let us calculate the vacuum energy to find the pair momentum $2\vec{q}$ or the angle β for a given parameters α_s, $\overline{\mu}$, and $\delta\mu$. We find the vacuum energy is given as in the HDL approximation $V(\Delta) \simeq -\Delta^3/(6\pi^3)\left(\mu_d q/\overline{\mu}\right)$. The best q or the angle β is determined by minimizing the vacuum energy for a given α_s. For $\alpha_s = 1$, we find $\cos\beta = 0.915$ and $V = -8.5\times 10^{-4}\alpha_s\overline{\mu}^4$. To find $\delta\mu_2$ for a given α_s, we minimize the vacuum energy with respect to β and $\delta\mu$. At the leading order the vacuum energy is minimized for large $\delta\mu$. Therefore, to determine the correct $\delta\mu_2$ we need to go beyond the current approximations and include the $\delta\mu/\overline{\mu}$ corrections in the gap equation. This result is consistent, at least qualitatively, with the observation in [15] that the LOFF window widens considerably in the one gluon exchange regime.

4 Patching

In the preceding analysis we assumed a simple plane wave condensate $\Delta(\vec{q}) \propto e^{2iq\cdot x}$. Kinematical constraints require that quarks which participate in this condensation lie on a ring-like region of volume $\sim \Delta^2\mu_i\sin\alpha_i$, where i = u,d. However, due to the rotaional invariance of the system any direction of the pair momentum $2\vec{q}$ is possible. Thus, one needs to patch condensates carrying same momentum but different direction to increase the binding energy The up and down quarks which pair in a LOFF condensate lie in rings of radius $\mu_i\sin\alpha_i$, where $i = u, d$. These rings lie in a plane perpendicular to $\vec{q}$, and have thickness of order Δ. If we sum over patches associated

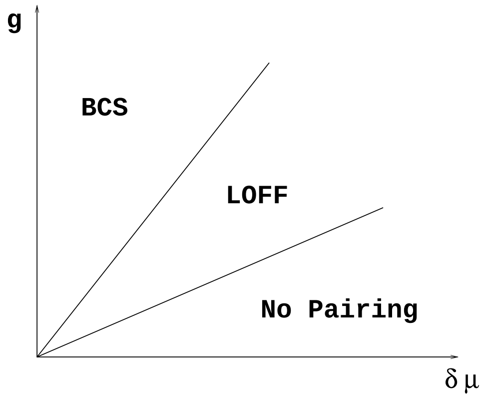

Figure 1: A (schematic) phase diagram in the coupling-$\delta\mu$ plane.

with planar rotations of $\vec{q}$ without overlap, we will recover a fraction Fermi surface which scales like μ^2 and thus LOFF phase can be compatible with BCS pairing.

5 Conclusions

We have used high density effective theory to study crystalline superconductivity, using a local four fermion interaction to model the interaction at intermediate density, and one gluon exchange for the asymptotic regime. We obtained analytic results for the condensate and vacuum energy, and conclude that the LOFF phase is quite plausibly favored $\Delta_0/\sqrt{2} < \delta\mu < \delta\mu_2$. We also discussed how disjoint regions of condensate could be patched together to obtain a lower vacuum energy. The optimal patching configuration has yet to be determined for generic β, but it seems possible that the binding energy of crystalline superconductivity scales as the area of the Fermi surface ($\sim \mu^2$) rather than as μ in the case of a plane wave LOFF condensate.

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Discussion

M. Mannarelli (INFN-Bari): You say that the LOFF phase appears for $g > g_c$ (in asymmetric quark matter). Is this the reason why in ordinary BCS it has not been observed a LOFF phase?

Hong: For a given coupling g, there is always a window for LOFF as $\delta\mu$ changes. However the window is larger for a stronger coupling. It is therefore much easier to find a LOFF phase in strongly coupled systems.

M. Alford (Glasgow University): Just to clarify, for any coupling strength there is a range of $\delta\mu$ in which LOFF pairing is favored. This means that since $\delta\mu$ is a function of r in a neutron star, there is a reasonable chance of seeing a shell of LOFF matter somewhere inside it.

Hong: For a given $\delta\mu$, the coupling between fermions has to be larger than g_c for a LOFF phase to exist. This is the point I was trying to emphasize during my talk. However, as the coupling becomes larger than g_{c1} ($> g_c$), the BCS gap becomes large enough to win against a LOFF phase. A schematic phase diagram[1] as a function of the coupling g and $\delta\mu$ can be given as Fig. 1:

[1] I thank M. Alford for helping me to understand the diagram in Fig. 1.