Effective description of QCD at high density

Roberto Casalbuoni
Dipartimento di Fisica dell'Universita'
Sezione INFN di Firenze
I-50125 Firenze, ITALY

1 Introduction

Ideas about color superconductivity go back to almost 25 years ago [1], but only recently this phenomenon has received a lot of attention (for recent reviews see refs. [2, 3]). The naive expectation is that at very high density, due to the asymptotic freedom, quarks would form a Fermi sphere of almost free fermions. However, Bardeen, Cooper and Schrieffer proved that the Fermi surface of free fermions is unstable in presence of an attractive, arbitrary small, interaction. Since in QCD the gluon exchange in the $\overline{3}$ channel is attractive one expects the formation of a coherent state of particle/hole pairs (Cooper pairs). An easy way to understand the origin of this instability is to remember that the Fermi energy distribution for free fermions at zero temperature is given by $f(E) = \theta(\mu - E)$ and therefore, the maximum value of the energy (Fermi energy) is $E_F = \mu$. Then consider the grand-potential $F = E - \mu N$. Adding or subtracting a particle (or adding a hole) to the Fermi surface does not change F, since $F \to (E \pm E_F) - \mu(N \pm 1) = F$. We see that the Fermi sphere of free fermions is highly degenerate. This is the origin of the instability, because if we add to the Fermi sphere two particles bounded with a binding energy E_B, the grand potential decreases, since $F - F_B = -E_B < 0$. Therefore in presence of an arbitrarily small attractive interaction, it is energetically more favorable for fermions to pair and form condensates.

We will discuss the methods for the quantitative evaluation of the condensates via a toy model. We will present also the results for the 2SC and CFL phases. Then we will introduce the effective lagrangian describing the low energy degrees of freedom of the CFL phase, that is the Goldstone bosons associated to the global symmetry breaking. We will then show how to formulate in a convenient way the idea of quasi-particles near the Fermi surface, introducing a formalism similar to the one used in heavy quark effective theory [4]. Next we will couple the quasi-particles to the Goldstone bosons in such a way to obtain the right gap term in the fermionic lagrangian. Using a weak coupling expansion, justified by asymptotic freedom, we will be able to evaluate the self-energy of the Goldstone bosons. The expansion at low momenta will allow us to get some of the couplings appearing in the low-energy effective lagrangian. The remaining couplings are then obtained through an analogous evaluation of the gluon self-energy.

2 Color condensation

Thee physics of fermions at finite density and zero temperature can be treated in a systematic way by using Landau's idea of quasi-particles. An example is the Landau theory of Fermi liquids. A conductor is treated as a gas of almost free electrons. However these electrons are dressed ones, where the dressing takes into account the interactions which are neglected in this description. According to Polchinski [5] this procedure just works because the interactions can be integrated away in the usual sense of the effective theories. Of course, this is a consequence of the special nature of the Fermi surface, which is such that there are practically no relevant or marginal interactions. In fact, all the interactions are irrelevant except for the four-fermi couplings between pairs of opposite momentum. Quantum corrections make the attractive ones relevant, and the repulsive ones irrelevant. This explains the instability of the Fermi surface of free fermions against attractive four-fermi interactions, but we would like to understand better the physics underlying the formation of the condensates and how the idea of quasi-particles comes about. To this purpose we will make use of a toy model involving two Fermi oscillators describing, for instance, spin up and spin down. Of course, in a finite-dimensional system there is no spontaneous symmetry breaking, but this model is useful just to illustrate the main points. We assume our dynamical system to be described by the following Hamiltonian containing a quartic coupling between the oscillators

$$H = \epsilon(a_1^\dagger a_1 + a_2^\dagger a_2) + G a_1^\dagger a_2^\dagger a_1 a_2. \tag{1}$$

We will study this model by using a variational principle. We start introducing the following normalized trial wave-function $|\Psi\rangle$

$$|\Psi\rangle = \left(\cos\theta + \sin\theta\, a_1^\dagger a_2^\dagger\right)|0\rangle. \tag{2}$$

The di-fermion operator, $a_1 a_2$, has the following expectation value

$$\Gamma \equiv \langle\Psi|\, a_1 a_2 |\Psi\rangle = -\sin\theta\cos\theta. \tag{3}$$

Then we determine the value of θ by looking for the minimum of the expectation value of H on the trial state

$$\langle\Psi|H|\Psi\rangle = 2\epsilon\sin^2\theta - G\Gamma^2. \tag{4}$$

We get

$$2\epsilon \sin 2\theta + 2G\Gamma \cos 2\theta = 0 \longrightarrow \tan 2\theta = -\frac{G\Gamma}{\epsilon}. \tag{5}$$

By using the expression (3) for Γ we obtain the gap equation

$$\Gamma = -\frac{1}{2}\sin 2\theta = \frac{1}{2}\frac{G\Gamma}{\sqrt{\epsilon^2 + G^2\Gamma^2}}, \tag{6}$$

or

$$1 = \frac{1}{2}\frac{G}{\sqrt{\epsilon^2 + \Delta^2}}, \tag{7}$$

where $\Delta = G\Gamma$. Therefore the gap equation can be seen as the equation determining the ground state of the system, since it gives the value of the condensate. We can now introduce the idea of quasi-particles in this particular context. Let us write the hamiltonian H as the sum of two pieces, a quadratic one, plus another piece containing the interaction term defined in such a way to have zero expectation value on the ground state $|\Psi\rangle$. We have

$$H = H_0 + H_{\rm res}, \tag{8}$$

with

$$H_0 = \epsilon(a_1^\dagger a_1 + a_2^\dagger a_2) - G\Gamma(a_1 a_2 - a_1^\dagger a_2^\dagger), \tag{9}$$

and

$$H_{\rm res} = G(a_1^\dagger a_2^\dagger + \Gamma)\,(a_1 a_2 - \Gamma)\,, \tag{10}$$

where we have neglected a constant term $G\Gamma^2$. Let us now look for a transformation on the Fermi oscillator variables such that H_0 acquires a canonical form (Bogoliubov transformation). Such a transformation is

$$A_1 = a_1 \cos\theta - a_2^\dagger \sin\theta, \quad A_2 = a_1^\dagger \sin\theta + a_2 \cos\theta, \tag{11}$$

and we get

$$H_0 = (\epsilon - \sqrt{\epsilon^2 + \Delta^2}) + \sqrt{\epsilon^2 + \Delta^2}(A_1^\dagger A_1 + A_2^\dagger A_2). \tag{12}$$

Furthermore we can check that

$$A_{1,2}|\Psi\rangle = 0. \tag{13}$$

Therefore the operators $A_i^\dagger$ create out of the vacuum quasi-particles of energy

$$E = \sqrt{\epsilon^2 + \Delta^2}. \tag{14}$$

We see that the condensation gives rise to the fermionic energy gap, Δ. The Bogoliubov transformation realizes the dressing of the original operators a_i and $a_i^\dagger$ to the quasi-particle ones A_i and $A_i^\dagger$. Of course, the interaction is still present, but part of it has been absorbed in the dressing process, and the hope is that in such a way one is able to get a better starting point for a perturbative expansion. As we have said this point of view has been very fruitful in the Landau theory of conductors.

Let us now discuss what have been done in order to determine the gap in QCD at high density (see [2]). At asymptotically high-density one can, in principle, perform the calculation starting from first principles, since QCD is weakly coupled [6] (for a more complete list of references see [2]). However the actual calculations are unlikely to be extrapolated below a chemical potential of order $10^8\ GeV$ [7]. Since the interesting density regime for neutron stars and heavy ions is for $\mu \leq 500\ MeV$, one has to use phenomenological interactions known to capture the essential features of QCD. In the case of two massless flavors one can use the instanton vertex producing a four-fermi interaction. For more flavors use has been made of the one-gluon exchange approximation or, again, of a four-fermi interaction with color, flavor, and spin structure identical to the one gluon exchange case. A practical way to arrive at the gap equation is to write down the Schwinger-Dyson equation for the case at hand, ignoring vertex corrections. One gets an equation of the type

$$\Sigma(k) = -\frac{1}{2\pi^4}\int d^4q\, G^{-1}(q)V(k-q), \tag{15}$$

where $G^{-1}(q)$ is the full propagator and $V(p)$ is the vertex function, momentum independent for a point-like interactions. This is an integral equation for the self-energy which appears on the right-hand side in $G^{-1}(q)$. Since one expects a diquark condensate it is convenient to introduce the Nambu-Gorkov fields

$$\Psi = \begin{pmatrix} \psi \\ \overline{\psi}^T \end{pmatrix}, \tag{16}$$

The wave operator is then postulated of the form

$$G(q) = G(q)_{\rm free} + \Sigma = \begin{pmatrix} \not{q} + \mu\gamma_0 & \gamma_0\Delta\gamma_0 \\ \Delta & (\not{q} - \mu\gamma_0)^T \end{pmatrix}, \tag{17}$$

equivalent to introduce in the lagrangian a mass gap term of the type $\psi^T C\Delta\psi$. For a four-fermi interaction Δ is a matrix in color-flavor-spin space. In the gluon-exchange case there is also a momentum dependence. The typical gap equation for a four-fermi interaction (neglecting antiparticle contribution) is

$$\Delta = 4G\int_0^\Lambda \frac{d^4p}{(2\pi)^4}\Big(\frac{\Delta}{p_0^2 + (|\vec{p}| - \mu)^2 + \Delta^2}\Big). \tag{18}$$

When the interaction strength, $G \to 0$, and $\Delta \ll \mu$, Λ, one finds

$$\Delta \propto \Delta\mu^2\, G\log(\mu/\Delta) \to \Delta \propto \mu\exp(-c/(G\mu^2)). \tag{19}$$

Replacing the effective four-fermi interaction with one-gluon exchange one expects

$$\Delta \propto \mu \exp(-c/g^2). \tag{20}$$

But taking into account the gluon propagator one gets a double-log contribution due to a collinear infrared divergence arising from the gluon propagator. This divergence is regulated, for $q \to 0$, by the inverse of the penetration length, which turns out to be of order Δ [8]. In the weak coupling limit one gets [6]

$$\Delta \propto \Delta g^2 (\log(\mu/\Delta))^2 \to \Delta \propto \mu \exp(-c/g). \tag{21}$$

The gap at large μ is much larger in QCD than in the case of a point-like interaction. If the coupling g is evaluated at μ and we assume $1/g^2 \approx \log \mu$ the exponential gives a very weak suppression and for $\mu \to \infty$ we have $\Delta \to \infty$ and $\Delta/\mu \to 0$. Color superconductivity is bounded to dominate physics at high density.

The phase structure of QCD at high density depends on the number of flavors and there are two very interesting cases, corresponding to two massless flavors (2SC) [1, 5] and to three massless flavors (CFL) [10, 11] respectively. In this talk we will be mainly concerned with the latter case. The two cases correspond to very different patterns of symmetry breaking. If we denote left- and right-handed quark fields by $q^\alpha_{iL(R)}$ with $\alpha = 1, 2, 3$, the $SU(3)_c$ color index, and $i = 1, \cdots, N_f$ the flavor index (N_f is the number of massless flavors), in the 2SC phase the previous calculations give the following color-flavor-spin structure for the condensate

$$\langle q^\alpha_{iL(R)}(\vec{p}) C q^\beta_{jL(R)}(-\vec{p})\rangle = \frac{\Delta}{G}\epsilon_{ij}\epsilon^{\alpha\beta 3}. \tag{22}$$

Here $C = i\gamma^2\gamma^0$ is the charge-conjugation matrix. The resulting symmetry breaking pattern (barring $U(1)$ factors) is [1, 5]

$$\begin{array}{c} SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \\ \downarrow \\ SU(2)_c \otimes SU(2)_L \otimes SU(2)_R \end{array} \tag{23}$$

The color group $SU(3)_c$ breaks down to the subgroup $SU(2)_c$ but no global symmetry is broken. Although the baryon number, B, is broken, there is a combination of B and of the broken color generator, T_8, which is unbroken in the 2SC phase. Therefore no massless Goldstone bosons are present in this phase. On the other hand, five gluon fields acquire mass whereas three are left massless. It is worth to notice that for the electric charge the situation is very similar to the one for the baryon number. Again a linear combination of the broken electric charge and of the broken generator T_8 is unbroken in the 2SC phase. The condensate (22) gives rise to a gap, Δ, for quarks of color 1 and 2, whereas the two quarks of color 3 remain un-gapped (massless). The resulting effective low-energy theory has been described in [12].

Numerically, by choosing a cutoff $\Lambda = 800\ MeV$ (see eq. (18)), and fixing the coupling G in such a way to reproduce correctly the physics at zero density and temperature, one finds [2]

$$\begin{aligned} \mu &= 400\ MeV \Rightarrow \Delta = 106\ MeV, \\ \mu &= 500\ MeV \Rightarrow \Delta = 145\ MeV. \end{aligned} \tag{24}$$

In this contribution we will be mainly interested in the formulation of the effective theory for three massless quarks. An analysis similar to the previous one for this case leads to the condensate [10, 11]

$$\langle q^i_{\alpha L}(\vec{p}) C q^j_{\beta\,L}(-\vec{p})\rangle = \frac{1}{G} P^{ij}_{\alpha\beta}, \tag{25}$$

with

$$P^{ij}_{\alpha\beta} = \frac{1}{3}(\Delta_8 + \frac{1}{8}\Delta_1)\delta^i_\alpha \delta^j_\beta + \frac{1}{8}\Delta_1 \delta^i_\beta \delta^j_\alpha. \tag{26}$$

This originates the breaking

$$SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_B \otimes U(1)_A$$

$$\downarrow \tag{27}$$

$$SU(3)_{c+L+R} \otimes Z_2 \otimes Z_2 \tag{28}$$

The $U(1)_A$ symmetry is broken at the quantum level by the anomaly, but it gets restored at very high density since the instanton contribution is suppressed [13, 14, 15]. Again, for $\Lambda = 800\ MeV$, a convenient choice for G and $\mu = 400\ MeV$, one gets [2]

$$\Delta_8 = 80\ MeV, \quad \Delta_1 = -176\ MeV. \tag{29}$$

The condensate can also be written in the form

$$\langle q^\alpha_{iL(R)}(\vec{p}) C q^\beta_{jL(R)}(-\vec{p})\rangle \propto \epsilon^{ijX}\epsilon_{\alpha\beta X} + \kappa(\delta^i_\alpha \delta^j_\beta + \delta^i_\beta \delta^j_\alpha). \tag{30}$$

Notice that $\kappa = 0$ corresponds to $\Delta_1 = -2\Delta_8$. Due to the Fermi statistics, the condensate must be symmetric in color and flavor. As a consequence the two terms appearing in eq. (30) correspond to the $(\bar{\mathbf{3}}, \bar{\mathbf{3}})$ and $(\mathbf{6}, \mathbf{6})$ channels of $SU(3)_c \otimes SU(3)_{L(R)}$. It turns out that κ is small [10, 14, 16] and therefore the condensation occurs mainly in the $(\bar{\mathbf{3}}, \bar{\mathbf{3}})$ channel (see also eq. (29)). The expression (30) shows that the ground state is left invariant by a simultaneous transformation of $SU(3)_c$ and $SU(3)_{L(R)}$. This is called Color Flavor Locking (CFL). The Z_2 symmetries arise since the condensate is left invariant by a change of sign of the left- and/or right-handed fields. As for the

2SC case the electric charge is broken but a linear combination with the broken color generator T_8 annihilates the ground state. On the contrary the baryon number is broken. Therefore there are $8+2$ broken global symmetries giving rise to 10 Goldstone bosons. The one associated to $U(1)_A$ gets massless only at very high density. The color group is completely broken and all the gauge particles acquire mass. Also all the fermions are gapped.

3 Effective theory for the CFL phase

We start introducing the Goldstone fields as the phases of the condensates in the $(\overline{\mathbf{3}},\overline{\mathbf{3}})$ channel [17, 18]

$$X^i_\alpha \approx \epsilon^{ijk}\epsilon_{\alpha\beta\gamma}\langle q^j_{\beta L} q^k_{\gamma L}\rangle^*, \quad Y^i_\alpha \approx \epsilon^{ijk}\epsilon_{\alpha\beta\gamma}\langle q^j_{\beta L} q^k_{\gamma L}\rangle^*. \tag{31}$$

Since quarks belong to the representation $(\mathbf{3},\mathbf{3})$ of $SU(3)_c\otimes SU(3)_{L(R)}$ and transform under $U(1)_B\otimes U(1)_A$ according to

$$qq_L \to e^{i(\alpha+\beta)}q_L, \quad q_R \to e^{i(\alpha-\beta)}q_R, \quad e^{i\alpha}\in U(1)_B, \quad e^{i\beta}\in U(1)_A, \tag{32}$$

the transformation properties of the fields X and Y under the total symmetry group $G=SU(3)_c\otimes SU(3)_L\otimes SU(3)_R\otimes U(1)_B\otimes U(1)_A$ are ($g_c\in SU(3)_c$, $g_{L(R)}\in SU(3)_{L(R)}$)

$$X\to g_c X g_L^T e^{-2i(\alpha+\beta)}, \quad Y\to g_c Y g_R^T e^{-2i(\alpha-\beta)}. \tag{33}$$

The fields X and Y are $U(3)$ matrices and as such they describe $9+9=18$ fields. Eight of these fields are eaten up by the gauge bosons, producing eight massive gauge particles. Therefore we get the right number of Goldstone bosons, $10=18-8$. These fields correspond to the breaking of the global symmetries in G (18 generators) to the symmetry group of the ground state $H=SU(3)_{c+L+R}\otimes Z_2\otimes Z_2$ (8 generators). For the following it is convenient to separate the $U(1)$ factors in X and Y in order to get new fields belonging to $SU(3)$

$$X=\hat{X}e^{2i(\phi+\theta)}, \quad Y=\hat{Y}e^{2i(\phi-\theta)}, \quad \hat{X},\hat{Y}\in SU(3). \tag{34}$$

The fields ϕ and θ can be also described in an a more invariant way through the determinants of X and Y

$$d_X=\det(X)=e^{6i(\phi+\theta)}, \quad d_Y=\det(Y)=e^{6i(\phi-\theta)}, \tag{35}$$

with transformation properties under G

$$\hat{X}\to g_c\hat{X}g_L^T, \quad \hat{Y}\to g_c\hat{Y}g_R^T, \quad \phi\to\phi-\alpha, \quad \theta\to\theta-\beta. \tag{36}$$

The breaking of the global symmetry can be discussed in terms of gauge invariant fields d_X, d_Y and

$$\Sigma_j^i = \sum_\alpha (\hat{Y}_\alpha^j)^* \hat{X}_\alpha^i \to \Sigma = \hat{Y}^\dagger \hat{X}. \tag{37}$$

The Σ field describes the 8 Goldstone bosons corresponding to the breaking of the chiral symmetry $SU(3)_L \otimes SU(3)_R$, as it is made clear by the transformation properties of Σ^T,

$$\Sigma^T \to g_L \Sigma^T g_R^\dagger. \tag{38}$$

That is Σ^T transforms exactly as the usual chiral field. The other two fields d_X and d_Y provide the remaining two Goldstone bosons corresponding to the breaking of the $U(1)$ factors.

In order to build up an invariant lagrangian it is convenient to define the following currents

$$J_X^\mu = \hat{X} D^\mu \hat{X}^\dagger = \hat{X}(\partial^\mu \hat{X}^\dagger + \hat{X}^\dagger g^\mu), \quad J_Y^\mu = \hat{Y} D^\mu \hat{Y}^\dagger = \hat{Y}(\partial^\mu \hat{Y}^\dagger + \hat{Y}^\dagger g^\mu), \tag{39}$$

with $g_\mu = i g_s g_\mu^a T^a/2$ the gluon field and $T^a = \lambda_a/2$ the $SU(3)_c$ generators. These currents have simple transformation properties under the full symmetry group G,

$$J_{X,Y}^\mu \to g_c J_{X,Y}^\mu g_c^\dagger. \tag{40}$$

The most general lagrangian, up to two derivative terms, invariant under G, the rotation group $O(3)$ (Lorentz invariance is broken by the chemical potential term) and the parity transformation defined as

$$\hat{X}(\vec{x},t) \leftrightarrow \hat{Y}(-\vec{x},t), \quad \phi(\vec{x},t) \to \phi(-\vec{x},t), \quad \theta(\vec{x},t) \to -\theta(-\vec{x},t), \tag{41}$$

is [17]

$$\begin{aligned}\mathcal{L} = & -\frac{F_T^2}{4}\mathrm{Tr}\left[(J_X^0 - J_Y^0)^2)\right] - \alpha_T \frac{F_T^2}{4}\mathrm{Tr}\left[(J_X^0 + J_Y^0)^2)\right] + \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\partial_0\theta)^2 \\ & + \frac{F_S^2}{4}\mathrm{Tr}\left[\left(\vec{J}_X - \vec{J}_Y)^2\right)\right] + \alpha_S \frac{F_S^2}{4}\mathrm{Tr}\left[\left(\vec{J}_X + \vec{J}_Y)^2\right)\right] - \frac{v_\phi^2}{2}|\vec{\nabla}\phi|^2 - \frac{v_\theta^2}{2}|\vec{\nabla}\theta|^2.\end{aligned} \tag{42}$$

Using $SU(3)_c$ color gauge invariance we can choose $\hat{X} = \hat{Y}^\dagger$, making 8 of the Goldstone bosons disappear and giving mass to the gluons. The properly normalized Goldstone bosons, Π^a, are given in this gauge by

$$\hat{X} = \hat{Y}^\dagger = e^{i\Pi^a T^a/F_T}. \tag{43}$$

Expanding eq. (42) at the lowest order in the fields we get

$$\mathcal{L} \approx \frac{1}{2}(\partial_0 \Pi^a)^2 + \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\partial_0\theta)^2 - \frac{v^2}{2}|\vec{\nabla}\Pi^a|^2 - \frac{v_\phi^2}{2}|\vec{\nabla}\phi|^2 - \frac{v_\theta^2}{2}|\vec{\nabla}\theta|^2, \tag{44}$$

with $v = F_s/F_T$. The gluons g_0^a and g_i^a acquire Debye and Meissner masses given by

$$m_D^2 = \alpha_T g_s^2 F_T^2, \quad m_M^2 = \alpha_S v^2 g_s^2 F_T^2. \tag{45}$$

It should be stressed that these are not the true rest masses of the gluons, since there is a large wave function renormalization effect making the gluon masses of the order of the gap Δ (see later) [8]. Since this description is supposed to be valid at low energies (we expect much below the gap Δ), we could also decouple the gluons solving their classical equations of motion neglecting the kinetic term. The result from eq. (42) is

$$g_\mu = -\frac{1}{2}\left(\hat{X}\partial_\mu \hat{X}^\dagger + \hat{Y}\partial_\mu \hat{Y}^\dagger\right). \tag{46}$$

It is easy to show that substituting this expression in eq. (42) one gets [8]

$$\mathcal{L} = \frac{F_T^2}{4}\left(\mathrm{Tr}[\dot{\Sigma}\dot{\Sigma}^\dagger] - v^2\mathrm{Tr}[\vec{\nabla}\Sigma\cdot\vec{\nabla}\Sigma^\dagger]\right) + \frac{1}{2}\left(\dot{\phi}^2 - v_\phi^2|\vec{\nabla}\phi|^2\right) + \frac{1}{2}\left(\dot{\theta}^2 - v_\phi^2|\vec{\nabla}\theta|^2\right). \tag{47}$$

The first term is nothing but the chiral lagrangian except for the breaking of the Lorentz invariance. This is a way of seeing the quark-hadron continuity, that is the continuity between the CFL and the nuclear matter in three flavor QCD. The identification is perfect if one realizes that in nuclear matter the pairing may occur in such a way to give rise to a superfluid due to the breaking of the baryon number as it happens in CFL [19].

4 Fermions near the Fermi surface

We will introduce now the formalism described in ref. [20] in order to evaluate several quantities of interest appearing in the effective lagrangian. This formulation is based on the observation that, at very high-density, the energy spectrum of a massless fermion is described by states $|\pm\rangle$ with energies $E_\pm = -\mu \pm |\vec{p}|$ where μ is the quark number chemical potential. For energies much lower than the Fermi energy μ, only the states $|+\rangle$ close to the Fermi surface. i.e. with $|\vec{p}\,| \approx \mu$, can be excited. On the contrary, the states $|-\rangle$ have $E_- \approx -2\mu$ and therefore they decouple.

This can be seen more formally by writing the four-momentum of the fermion as

$$p^\mu = \mu v^\mu + \ell^\mu, \tag{48}$$

where $v^\mu = (0, \vec{v}_F)$, and $\vec{v}_F$ is the Fermi velocity defined as $\vec{v}_F = \partial E/\partial\vec{p}|_{\vec{p}=\vec{p}_F}$. For massless fermions $|\vec{v}_F| = 1$. Since the hamiltonian for a massless Dirac fermion in a chemical potential μ is

$$H = -\mu + \vec{\alpha}\cdot\vec{p}, \quad \vec{\alpha} = \gamma_0\vec{\gamma}, \tag{49}$$

one has

$$H = -\mu(1 - \vec{\alpha}\cdot\vec{v}_F) + \vec{\alpha}\cdot\vec{\ell}. \tag{50}$$

Then, it is convenient to introduce the projection operators

$$P_\pm = \frac{1 \pm \vec{\alpha}\cdot\vec{v}_F}{2}, \tag{51}$$

such that

$$H|+\rangle = \vec{\alpha}\cdot\vec{\ell}\,|+\rangle, \quad H|-\rangle = (-2\mu + \vec{\alpha}\cdot\vec{\ell})\,|-\rangle. \tag{52}$$

We can define fields corresponding to the states $|\pm\rangle$ through the decomposition

$$\psi(x) = \sum_{\vec{v}_F} e^{-i\mu v\cdot x}\left[\psi_+(x) + \psi_-(x)\right], \tag{53}$$

where an average over the Fermi velocity $\vec{v}_F$ is performed. The velocity-dependent fields $\psi_\pm(x)$ are given by ($v^\mu = (0, \vec{v}_F)$)

$$\psi_\pm(x) = e^{i\mu v\cdot x}\left(\frac{1 \pm \vec{\alpha}\cdot\vec{v}_F}{2}\right)\psi(x) \;=\; \int_{|\ell|<\delta}\frac{d^4\ell}{(2\pi)^4}e^{-i\,\ell\cdot x}\psi_\pm(\ell). \tag{54}$$

Since we are interested at physics near the Fermi surface we integrate out all the modes with $|\ell| > \delta$, where δ is a cut-off such that $\delta \ll \mu$. At the same time we will choose δ greater than the energy gap, Δ. Since, as we shall see, physics around the Fermi surface is effectively two-dimensional, the results do not depend on the value of δ. Substituting inside the Dirac part of the QCD lagrangian density one obtains ($V^\mu = (1, \vec{v}_f)$, $\tilde{V}^\mu = (1, -\vec{v}_F)$)

$$\mathcal{L} = \sum_{\vec{v}_F}\Big[\psi_+^\dagger iV\cdot D\psi_+ + \psi_-^\dagger(2\mu + i\tilde{V}\cdot D)\psi_- + (\overline{\psi}_+ i\,\not{D}_\perp\psi_- + \text{h.c.})\Big], \tag{55}$$

where $\not{D}_\perp = D_\mu\gamma_\perp^\mu$ and

$$\gamma_\perp^\mu \;=\; \frac{1}{2}\gamma_\nu\left(2g^{\mu\nu} - V^\mu\tilde{V}^\nu - \tilde{V}^\mu V^\nu\right). \tag{56}$$

We notice that the fields appearing in this expression are evaluated at the same Fermi velocity because off-diagonal terms are cancelled by the rapid oscillations of the exponential factor in the $\mu \to \infty$ limit. This behaviour can be referred to as the Fermi velocity superselection rule.

At the leading order in $1/\mu$ one has

$$iV\cdot D\psi_+ = 0, \qquad \psi_- = -\frac{i}{2\mu}\gamma_0\,\not{D}_\perp\psi_+, \tag{57}$$

showing the decoupling of ψ_- for $\mu\to\infty$. The equation for ψ_+ shows also that only the energy and the momentum parallel to the Fermi velocity are relevant variables in the problem. We have an effective two-dimensional theory.

Eliminating the field ψ_- we get

$$\mathcal{L}=\sum_{\vec v_F}\left[\psi_+^\dagger iV\cdot D\psi_+-\frac{1}{2\mu+i\tilde V\cdot D}\,\psi_+^\dagger(\not{D}_\perp)^2\psi_+\right]. \tag{58}$$

The previous remarks apply to any theory describing massless fermions at high density. The next step will be to couple this theory in a $SU(3)_L\otimes SU(3)_R\otimes SU(3)_c$ invariant way to Nambu-Goldstone bosons (NGB) describing the appropriate breaking for the CFL phase. Using a gradient expansion we get an explicit expression for the decay coupling constant of the Nambu-Goldstone bosons as well for their velocity.

5 Goldstone bosons self-energy

The invariant coupling between fermions and Goldstone fields reproducing the symmetry breaking pattern of eq. (30) is proportional to

$$\gamma_1\,Tr[\psi_L^T\hat X^\dagger]\,C\,Tr[\psi_L\hat X^\dagger]+\gamma_2\,Tr[\psi_L^T C\hat X^\dagger\psi_L\hat X^\dagger]+\text{h.c.}, \tag{59}$$

and analogous relations for the right-handed fields. Here the spinors are meant to be Dirac spinors. The trace is operating over the group indices of the spinors and of the Goldstone fields. Since the vacuum expectation value of the Goldstone fields is $\langle\hat X\rangle=\langle\hat Y\rangle=1$, we see that this coupling induces the right breaking of the symmetry. In the following we will consider only the case $\gamma_2=-\gamma_1\propto\Delta/2$, where Δ is the gap parameter.

Since the transformation properties under the symmetry group of the fields at fixed Fermi velocity do not differ from those of the quark fields, for both left-handed and right-handed fields we get the effective lagrangian density [8]

$$\begin{aligned}\mathcal{L} &= \sum_{\vec v_F}\frac{1}{2}\Big[\sum_{A=1}^{9}\Big(\psi_+^{A\dagger}iV\cdot D\psi_+^A+\psi_-^{A\dagger}i\tilde V\cdot D\psi_-^A-\Delta_A\Big(\psi_-^{A^T}C\psi_+^A+\text{h.c.}\Big)\Big)\\ &- \Delta\sum_{I=1,3}\Big(Tr[(\psi_-X_1^\dagger)^T C\epsilon_I(\psi_+X_1^\dagger)\epsilon_I]+\text{h.c.}\Big)\Big],\end{aligned} \tag{60}$$

where we have introduced the fields $\psi_\pm^A$:

$$\psi_\pm=\frac{1}{\sqrt2}\sum_{A=1}^{9}\lambda_A\psi_\pm^A. \tag{61}$$

Here λ_a $(a = 1, ..., 8)$ are the Gell-Mann matrices normalized as follows: $Tr(\lambda_a\lambda_b) = 2\delta_{ab}$ and $\lambda_9 = \sqrt{2/3}\,\mathbf{1}$. Furthermore $\Delta_1 = \cdots = \Delta_8 = \Delta$, $\Delta_9 = -2\Delta$, and $X_1 = \hat{X} - 1$. Notice that the NGB fields couple to fermionic fields with opposite Fermi velocities. In this expression, as in the following ones, the field ψ_- is defined as ψ_+ with $\vec{v}_F \to -\vec{v}_F$, and therefore it is not the same as the one defined in (54).

The formalism becomes more compact by introducing the Nambu-Gorkov fields

$$\chi = \begin{pmatrix} \psi_+ \\ C\psi_-^* \end{pmatrix}. \tag{62}$$

It is important to realize that the fields χ and $\chi^\dagger$ are not independent variables. In fact, since we integrate over all the Fermi surface, the fields ψ_-^* and ψ_+, appearing in χ, appear also in $\chi^\dagger$ when $\vec{v}_F \to -\vec{v}_F$. In order to avoid this problem we can integrate over half of the Fermi surface, or, taking into account the invariance under $\vec{v}_F \to -\vec{v}_F$, we can simply integrate over all the sphere with a weight $1/8\pi$ instead of $1/4\pi$:

$$\sum_{\vec{v}_F} = \int \frac{d\vec{v}_F}{8\pi}. \tag{63}$$

Then the first three terms in the lagrangian density (60) become

$$\mathcal{L}_0 = \int \frac{d\vec{v}_F}{8\pi}\,\frac{1}{2}\sum_{A=1}^{9} \chi^{A\dagger} \begin{bmatrix} iV\cdot D & \Delta^A \\ \Delta^A & i\tilde{V}\cdot D^* \end{bmatrix} \chi^A, \tag{64}$$

so that, in momentum space the free fermion propagator is

$$S_{AB}(p) = \frac{2\delta_{AB}}{V\cdot p\,\tilde{V}\cdot p - \Delta_A^2} \begin{bmatrix} \tilde{V}\cdot p & -\Delta_A \\ -\Delta_A & V\cdot p \end{bmatrix}. \tag{65}$$

We are now in position to evaluate the self-energy of the Goldstone bosons through their couplings to the fermions at the Fermi surface. There are two one-loop contributions [8], one from the coupling $\Pi\chi\chi$ and a tadpole from the coupling $\Pi\Pi\chi\chi$ arising from eq. (60). The tadpole diagram contributes only to the mass term and it is essential to cancel the external momentum independent term arising from the other diagram. Therefore, as expected, the mass of the NGB's is zero. The contribution at the second order in the momentum expansion is given by

$$i\,\frac{21 - 8\ln 2}{72\pi^2 F_T^2} \int \frac{d\vec{v}_F}{4\pi} \sum_{a=1}^{8} \Pi^a\, V\cdot p\,\tilde{V}\cdot p\,\Pi^a. \tag{66}$$

Integrating over the velocities and going back to the coordinate space we get

$$\mathcal{L}_{\rm eff}^{\rm kin} = \frac{21 - 8\ln 2}{72\pi^2 F_T^2} \sum_{a=1}^{8} \left(\dot{\Pi}^a\dot{\Pi}^a - \frac{1}{3}|\vec{\nabla}\Pi_a|^2 \right). \tag{67}$$

We can now determine the decay coupling constant F_T through the requirement of getting the canonical normalization for the kinetic term; this implies

$$F_T^2 = \frac{\mu^2(21 - 8\ln 2)}{36\pi^2}, \tag{68}$$

a result obtained by many authors using different methods (for a complete list of the relevant papers see the first reference of [2]). We see also that $v^2 = 1/3$. The origin of the pion velocity $1/\sqrt{3}$ is a direct consequence of the integration over the Fermi velocity. Therefore it is completely general and applies to all the NGB's in the theory, including the ones associated to the breaking of $U(1)_V$ and $U(1)_A$ ($v_\phi^2 = v_\theta^2 = 1/3$); needless to say, higher order terms in the expansion $1/\mu$ could change this result.

The breaking of the Lorentz invariance exhibited by the pion velocity different from one, can be seen also in the matrix element $\langle 0|J_\mu^a|\Pi^b\rangle$. Its evaluation gives [8]

$$\langle 0|J_\mu^a|\Pi^b\rangle = iF_T\delta_{ab}\tilde{p}_\mu, \qquad \tilde{p}^\mu = (p^0, \vec{p}/3). \tag{69}$$

The current is conserved, as a consequence of the dispersion relation satisfied by the NGB's.

6 Gluons self-energy

The other couplings appearing in our effective lagrangian can be obtained via a direct calculation of m_D and m_M [8]. This is done evaluating the one-loop contribution to the gluon self-energy. Also in this case there are two contributions, one coming from the gauge coupling to the fermions, whereas the other arises from the second term (sea-gull like) appearing in the fermion effective lagrangian of eq. (58). The results we find are [8]

$$m_D^2 = g_s^2 F_T^2, \quad m_M^2 = \frac{1}{3} m_D^2. \tag{70}$$

Comparison with equation (45) shows that

$$\alpha_S = \alpha_T = 1. \tag{71}$$

Performing a gradient expansion of the gluon self-energy one finds that there is a wave function renormalization of order $g_s\mu/\Delta \gg 1$ [8]. In fact, considering the different components of the gluon field, the temporal, g_0^a and the longitudinal, $g_L^{i\,a}$, and transverse, $g_T^{i\,a}$ ones, defined as

$$\begin{aligned} g_L^{i\,a} &= \frac{\vec{p}\cdot\vec{g}^a}{|\vec{p}|^2}\,p^i, \\ g_T^{i\,a} &= g^{i\,a} - g_L^{i\,a}\,, \end{aligned} \tag{72}$$

the following dispersion relations are obtained

$$
\begin{aligned}
g^{0a}: \quad 3\alpha_1\, E^2 - \alpha_1\, |\vec{p}\,|^2 &= m_D^2, \\
g_L^{ia}: \quad \alpha_1\, E^2 - \alpha_2\, \frac{|\vec{p}\,|^2}{3} &= m_M^2, \\
g_T^{ia}: \quad \alpha_1\, E^2 - \alpha_3\, |\vec{p}\,|^2 &= m_M^2,
\end{aligned}
\tag{73}
$$

where

$$
\begin{aligned}
\alpha_1 &= \frac{\mu^2 g_s^2}{216\Delta^2\pi^2}\left(7+\frac{16}{3}\ln 2\right), \\
\alpha_2 &= -\frac{\mu^2 g_s^2}{3240\Delta^2\pi^2}\left(59-\frac{688}{3}\ln 2\right), \\
\alpha_3 &= -\frac{\mu^2 g_s^2}{3240\Delta^2\pi^2}\left(41-\frac{112}{3}\ln 2\right) .
\end{aligned}
\tag{74}
$$

The coefficients of the energy give just the square of the wave function renormalization (notice that we have neglected the term coming from the free equation of motion, since the renormalization part is much bigger than one). If we extrapolate in momenta up to order Δ we see that the physical masses of the gluons turn out to be of the order of the gap energy ($\approx 1.70\Delta$) [8]. The validity of this extrapolation has been recently confirmed in ref. [21]. Wave function renormalization of order $g_s\mu/\Delta$ for gauge fields in a dense fermionic medium appears to be a rather general phenomenon. For instance, consider the 2SC phase. The low energy degrees of freedom are 3 gluons and the almost free quarks of color 3. The symmetries determining the effective lagrangian are: the gauge symmetry $SU(2)_c$ and rotation invariance (Lorentz is broken being at finite density). For the gluons one gets [22]

$$
\mathcal{L}_{\rm eff} = \frac{\epsilon}{2}\vec{E}^a\cdot\vec{E}^a - \frac{1}{2\lambda}\vec{B}^a\cdot\vec{B}^a, \tag{75}
$$

with a propagation velocity for the gluons given by $v = 1/\sqrt{\epsilon\lambda}$. Values of ϵ and λ different from 1 originate from wave function renormalization. One finds [22]

$$
\epsilon = 1 + \frac{g_s^2\mu^2}{18\pi^2\Delta^2} \approx \frac{g_s^2\mu^2}{18\pi^2\Delta^2}, \quad \lambda = 1. \tag{76}
$$

The strong coupling constant gets modified

$$
\alpha_s \;\to\; \alpha_s' = \frac{g_{\rm eff}^2}{4\pi v} = \frac{g_s^2}{4\pi\sqrt{\epsilon}} = \frac{3}{2\sqrt{2}}\frac{g_s\Delta}{\mu}, \tag{77}
$$

due to the changes in the propagation velocity and in the Coulomb force

$$
g_s^2/r \to g_s^2/(\epsilon r) \quad \Rightarrow \quad g_s^2 \to g_{\rm eff}^2 = g_s^2/\epsilon. \tag{78}
$$

Similar results hold for the massive gluons of type 4, 5, 6 and 7 which acquire a mass of order Δ. Exceptions are the spatial components (but not the time one) of the gluon 8. In this case there is no wave function renormalization of the time derivative and the mass is of order $g_s\mu$ [23]. Also the em dielectric constant gets modified by the in-medium effects both in the CFL and in the 2SC phases [24]

$$\tilde{\epsilon} = 1 + \frac{r}{18\pi^2}\frac{\tilde{e}^2\mu^2}{\Delta^2}, \tag{79}$$

where $\tilde{e}$ is the in-medium rotated electric charge, and

$$r = 4 \text{ in CFL}, \quad r = 1 \text{ in 2SC}. \tag{80}$$

Acknowledgments

I would like to thank R. Gatto, M. Mannarelli and G. Nardulli for their precious collaboration in the works originating this contribution.

References

[1] B. Barrois, Nucl. Phys. B **129**, 390 (1977); S. Frautschi, *Proceedings of workshop on hadronic matter at extreme density*, Erice 1978; D. Bailin and A. Love, Phys. Rep. **107** (1984) 325.

[2] K. Rajagopal and F. Wilczek, hep-ph/0011333;

[3] S.D.H. Hsu, hep-ph/0003140; D.K. Hong, Acta Phys. Pol. **B32** (2001) 1253, hep-ph/0101025; M. Alford, hep-ph/0102047.

[4] N. Isgur and M.B. Wise, Phys. Lett. **B232** (1989) 113 ; *ibidem* Phys. Lett. **B237** (1990) 527; E. Eichten and B. Hill, Phys. Lett. **B234** (1990) 511; H. Georgi, Phys. Lett. **B240** (1990) 447; for a recent review see A. V. Manohar and M.B. Wise *Heavy Quark Physics*, Cambridge University Press (2000).

[5] J. Polchinski, Lectures presented at TASI 92, hep-th/9210046.

[6] D. T. Son, Phys. Rev. D **59** (1999) 094019, hep-ph/9812287.

[7] K. Rajagopal and E. Shuster, Phys. Rev. D **62** (2000) 085007, hep-ph/0004074.

[8] R. Casalbuoni, R. Gatto and G. Nardulli, Phys. Lett. B **498** (2001) 179, hep-ph/0010321.

[9] M. Alford, K. Rajagopal and F. Wilczek, Phys. Lett. B **422**, 247 (1998), hep-ph/9711395; R. Rapp, T. Schäfer, E.V. Shuryak and M. Velkovsky, Phys. Rev. Lett. **81**, 53 (1998), hep-ph/9711396.

[10] M. Alford, K. Rajagopal and F. Wilczek, Nucl. Phys. B **537**, 443 (1999), hep-ph/9804403.

[11] T. Schäfer and F. Wilczek, Phys. Rev. Lett. **82**, 3956 (1999), hep-ph/9811473.

[12] R. Casalbuoni, Z. Duan and F. Sannino, Phys. Rev. D **62**, 094004 (2000), hep-ph/0004207; *ibidem* D **63**, 114026 (2001), hep-ph/0011394.

[13] R. Rapp, T. Schäfer , E.V. Shuryak and M. Velkovsky, Ann. of Phys. **280**, 35 (2000), hep-ph/9904353.

[14] T. Schäfer, Nucl. Phys. B **575**, 269 (2000), hep-ph/9909574.

[15] D.T. Son and M.A. Stephanov, Phys. Rev. D **61**, 074012 (2000), hep-ph/9910491; *ibidem* Erratum D **62**, 059902 (2000), hep-ph/0004095.

[16] I.A. Shovkovy and L.C. Wijewardhana, Phys. Lett. B **470**, 189 (1999), hep-ph/9910225.

[17] R. Casalbuoni and R. Gatto, Phys. Lett. B **464**, 111 (1999) , hep-ph/9908227.

[18] D.K. Hong, M. Rho and I. Zahed, Phys. Lett. B **468**, 261 (1999), hep-ph/9906551.

[19] T.Schäfer and F. Wilczek, Phys. Rev. Lett. **82**, 3956 (1999), hep-ph/9811473.

[20] D.K. Hong, Phys. Lett. B **473**, 118 (2000), hep-ph/9812510; D.K. Hong *Nuclear Physics* B **582**, 451 (2000), hep-ph/9905523; S.R. Beane, P.F. Bedaque and M.J. Savage, Phys. Lett. B **483**, 131 (2000), hep-ph/0002209.

[21] V. P. Gusynin and I. A. Shovkovy, hep-ph/0108175.

[22] D. H. Rischke, D. T. Son and M. A. Stephanov, Phys. Rev. Lett. **87** (2001) 062001, hep-ph/0011379.

[23] R. Casalbuoni, R. Gatto, M. Mannarelli and G. Nardulli, hep-ph/0107024.

[24] D. F. Litim and C. Manuel, hep-ph/0105165.