

On Classification of Irreducible Representations of q -Deformed Algebra $U'_q(\mathfrak{so}_n)$ Related to Quantum Gravity

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Classification of finite dimensional irreducible representations of nonstandard q -deformation $U'_q(\mathfrak{so}_n)$ of the universal enveloping algebra $U(\mathfrak{so}(n, \mathbb{C}))$ of the Lie algebra $\mathfrak{so}(n, \mathbb{C})$ (which does not coincide with the Drinfeld–Jimbo quantized universal enveloping algebra $U_q(\mathfrak{so}_n)$) is given for the case when q is not a root of unity. It is shown that such representations are exhausted by representations of the classical and nonclassical types. Examples of the algebras $U'_q(\mathfrak{so}_3)$ and $U'_q(\mathfrak{so}_4)$ are considered in detail. Notions of weights, highest weights, highest weight vectors are introduced. Raising and lowering operators for irreducible finite dimensional representations of $U'_q(\mathfrak{so}_n)$ are introduced. They depend on weight upon which they act. Explicit formulas for these operators are given.

1 Introduction

Quantum orthogonal groups, quantum Lorentz groups and their quantized universal enveloping algebras are of special interest for modern mathematics and physics. M. Jimbo [1] and V. Drinfeld [2] defined q -deformations (quantized universal enveloping algebras) $U_q(g)$ for all simple complex Lie algebras g by means of Cartan subalgebras and root subspaces (see also [3] and [4]). However, these approaches do not give a satisfactory presentation of the quantized algebra $U_q(\mathfrak{so}(n, \mathbb{C}))$ from a viewpoint of some problems in quantum physics and mathematics. Considering irreducible representations of the quantum groups $SO_q(n+1)$ and $SO_q(n, 1)$ we are interested in reducing them onto the quantum subgroup $SO_q(n)$. This reduction would give an analogue of the Gel'fand–Tsetlin basis for these representations. However, definitions of quantized universal enveloping algebras, mentioned above, do not allow the inclusions $U_q(\mathfrak{so}(n+1, \mathbb{C})) \supset U_q(\mathfrak{so}(n, \mathbb{C}))$ and $U_q(\mathfrak{so}(n, 1)) \supset U_q(\mathfrak{so}(n))$. To be able to exploit such reductions we have to consider q -deformation of the universal enveloping algebra of the Lie algebra $\mathfrak{so}(n+1, \mathbb{C})$ defined in terms of the generators $I_{k,k-1} = E_{k,k-1} - E_{k-1,k}$ (where E_{is} is the matrix with elements $(E_{is})_{rt} = \delta_{ir}\delta_{st}$) rather than by means of Cartan subalgebras and root elements. To construct such deformations we have to deform trilinear relations for elements $I_{k,k-1}$ instead of Serre's relations (used in the case of quantized universal enveloping algebras of Drinfeld and Jimbo). As a result, we obtain an associative algebra which will be denoted as $U'_q(\mathfrak{so}_n)$.

This q -deformation was first constructed in [5]. It permit us to construct the reductions of $U'_q(\mathfrak{so}_{n,1})$ and $U'_q(\mathfrak{so}_{n+1})$ onto $U'_q(\mathfrak{so}_n)$. The q -deformed algebra $U'_q(\mathfrak{so}_n)$ leads for $n = 3$ to the q -deformed algebra $U'_q(\mathfrak{so}_3)$ defined by D. Fairlie [6]. The cyclically symmetric algebra, similar to Fairlie's one, was also considered somewhat earlier by Odesskii [7]. The algebra $U'_q(\mathfrak{so}_4)$ is a q -deformation of the algebra $U(\mathfrak{so}(4, \mathbb{C}))$ given by means of commutation relations between the elements I_{ji} , $1 \leq i < j \leq 4$. For the Lie algebra $\mathfrak{so}(4, \mathbb{C})$ we have $\mathfrak{so}(4, \mathbb{C}) = \mathfrak{so}(3, \mathbb{C}) + \mathfrak{so}(3, \mathbb{C})$, while in the case of our q -deformation $U'_q(\mathfrak{so}_4)$ this is not the case (see e.g. [8]).

In the classical case, the imbedding $SO(n) \subset SU(n)$ (and its infinitesimal analogue) is of great importance for nuclear physics and in the theory of Riemannian symmetric spaces. It is

well known that in the framework of Drinfeld–Jimbo quantum groups and algebras one cannot construct the corresponding embedding. The algebra $U'_q(\mathfrak{so}_n)$ allows to define such an embedding [9], that it is possible to define the embedding $U'_q(\mathfrak{so}_n) \subset U_q(\mathfrak{sl}_n)$, where $U_q(\mathfrak{sl}_n)$ is a Drinfeld–Jimbo quantum algebra.

As a disadvantage of the algebra $U'_q(\mathfrak{so}_n)$ we have to mention the difficulties with Hopf algebra structure. Nevertheless, $U'_q(\mathfrak{so}_n)$ turns out to be a coideal in $U_q(\mathfrak{sl}_n)$ (see [9]) and this fact allows us to consider tensor products of finite dimensional irreducible representations of $U'_q(\mathfrak{so}_n)$ for many interesting cases (see [10]).

Finite dimensional irreducible representations of the algebra $U'_q(\mathfrak{so}_n)$ for q being not a root of unity were constructed in [5]. The formulas of action of the generators of $U'_q(\mathfrak{so}_n)$ upon the basis (which is a q -analogue of the Gel'fand–Tsetlin basis) are given there. A proof of these formulas and some their corrections were given in [11]. However, finite dimensional irreducible representations described in [5] and [11] are representations of the classical type. They are q -deformations of the corresponding irreducible representations of the Lie algebra \mathfrak{so}_n , that is, at $q \rightarrow 1$ they turn into representations of \mathfrak{so}_n .

If q is not a root of unity, the algebra $U'_q(\mathfrak{so}_n)$ has other classes of finite dimensional irreducible representations which have no classical analogues. These representations are singular at the limit $q \rightarrow 1$. They are described in [12]. A detailed description of these representations for the algebra $U'_q(\mathfrak{so}_3)$ is given in [13]. A classification of irreducible $*$ -representations of real forms of the algebra $U'_q(\mathfrak{so}_3)$ is given in [14].

The aim of this paper is to give classification theorem for finite dimensional irreducible representations of the algebra $U'_q(\mathfrak{so}_n)$ on complex vector spaces when q is not a root of unity. We show that in this case all irreducible finite dimensional representations of $U'_q(\mathfrak{so}_n)$ are exhausted by representations of the classical and nonclassical types. Detailed proofs of propositions and theorems, given in this paper, will be given separately.

Everywhere below we assume that q is not a root of unity.

2 Definition of the q -deformed algebra $U'_q(\mathfrak{so}_n)$

An existence of a q -deformation of the universal enveloping algebra $U(\mathfrak{so}(n, \mathbb{C}))$, different from the Drinfeld–Jimbo quantized universal enveloping algebra $U_q(\mathfrak{so}_n)$, is explained by the following reason. The Lie algebra $\mathfrak{so}(n, \mathbb{C})$ has two structures:

(a) The structure related to existing in $\mathfrak{so}(n, \mathbb{C})$ a Cartan subalgebra and root elements. A quantization of this structure leads to the Drinfeld–Jimbo quantized universal enveloping algebra $U_q(\mathfrak{so}_n)$.

(b) The structure related to realization of $\mathfrak{so}(n, \mathbb{C})$ by skew-symmetric matrices. In the Lie algebra $\mathfrak{so}(n, \mathbb{C})$ there exists a basis consisting of the matrices I_{ij} , $i > j$, defined as $I_{ij} = E_{ij} - E_{ji}$, where E_{ij} is the matrix with entries $(E_{ij})_{rs} = \delta_{ir}\delta_{js}$. These matrices are not root elements.

Using the structure (b), we may say that the universal enveloping algebra $U(\mathfrak{so}(n, \mathbb{C}))$ is generated by the elements I_{ij} , $i > j$. But in order to generate the universal enveloping algebra $U(\mathfrak{so}(n, \mathbb{C}))$, it is enough to take only the elements $I_{21}, I_{32}, \dots, I_{n, n-1}$. It is a minimal set of elements necessary for generating $U(\mathfrak{so}(n, \mathbb{C}))$. These elements satisfy the relations

$$\begin{aligned} I_{i, i-1}^2 I_{i+1, i} - 2I_{i, i-1} I_{i+1, i} I_{i, i-1} + I_{i+1, i} I_{i, i-1}^2 &= -I_{i+1, i}, \\ I_{i, i-1} I_{i+1, i}^2 - 2I_{i+1, i} I_{i, i-1} I_{i+1, i} + I_{i+1, i}^2 I_{i, i-1} &= -I_{i, i-1}, \\ I_{i, i-1} I_{j, j-1} - I_{j, j-1} I_{i, i-1} &= 0 \quad \text{for } |i - j| > 1. \end{aligned}$$

The following theorem is true [15] for the universal enveloping algebra $U(\mathfrak{so}(n, \mathbb{C}))$:

Theorem 1. *The universal enveloping algebra $U(\mathfrak{so}(n, \mathbb{C}))$ is isomorphic to the complex associative algebra (with a unit element) generated by the elements $I_{21}, I_{32}, \dots, I_{n,n-1}$ satisfying the above relations.*

We make the q -deformation of these relations by fulfilling the deformation of the integer 2 in these relations as

$$2 \rightarrow [2]_q := (q^2 - q^{-2}) / (q - q^{-1}) = q + q^{-1}.$$

As a result, we obtain the complex unital (that is, with a unit element) associative algebra generated by elements $I_{21}, I_{32}, \dots, I_{n,n-1}$ satisfying the relations

$$I_{i,i-1}^2 I_{i+1,i} - (q + q^{-1}) I_{i,i-1} I_{i+1,i} I_{i,i-1} + I_{i+1,i} I_{i,i-1}^2 = -I_{i+1,i}, \tag{1}$$

$$I_{i,i-1} I_{i+1,i}^2 - (q + q^{-1}) I_{i+1,i} I_{i,i-1} I_{i+1,i} + I_{i+1,i}^2 I_{i,i-1} = -I_{i,i-1}, \tag{2}$$

$$I_{i,i-1} I_{j,j-1} - I_{j,j-1} I_{i,i-1} = 0 \quad \text{for } |i - j| > 1. \tag{3}$$

This algebra was introduced by us in [5] and is denoted by $U'_q(\mathfrak{so}_n)$.

The analogue of the elements I_{ij} , $i > j$, can be introduced into $U'_q(\mathfrak{so}_n)$ (see [16]). In order to give them we use the notation $I_{k,k-1} \equiv I_{k,k-1}^+ \equiv I_{k,k-1}^-$. Then for $k > l + 1$ we define recursively

$$I_{kl}^+ := [I_{l+1,l}, I_{k,l+1}]_q \equiv q^{1/2} I_{l+1,l} I_{k,l+1} - q^{-1/2} I_{k,l+1} I_{l+1,l}, \tag{4}$$

$$I_{kl}^- := [I_{l+1,l}, I_{k,l+1}]_{q^{-1}} \equiv q^{-1/2} I_{l+1,l} I_{k,l+1} - q^{1/2} I_{k,l+1} I_{l+1,l}.$$

The elements I_{kl}^+ , $k > l$, satisfy the commutation relations

$$[I_{ln}^+, I_{kl}^+]_q = I_{kn}^+, \quad [I_{kl}^+, I_{kn}^+]_q = I_{ln}^+, \quad [I_{kn}^+, I_{ln}^+]_q = I_{kl}^+ \quad \text{for } k > l > n, \tag{5}$$

$$[I_{kl}^+, I_{nr}^+] = 0 \quad \text{for } k > l > n > r \quad \text{and } k > n > r > l, \tag{6}$$

$$[I_{kl}^+, I_{nr}^+]_q = (q - q^{-1}) (I_{lr}^+ I_{kn}^+ - I_{kr}^+ I_{nl}^+) \quad \text{for } k > n > l > r. \tag{7}$$

For I_{kl}^- , $k > l$, the commutation relations are obtained from these relations by replacing I_{kl}^+ by I_{kl}^- and q by q^{-1} .

The algebra $U'_q(\mathfrak{so}_n)$ can be defined as a unital associative algebra generated by I_{kl}^+ , $1 \leq l < k \leq n$, satisfying the relations (5)–(7). In fact, using the relations (4) we can reduce the relations (5)–(7) to the relations (1)–(3) for $I_{21}, I_{32}, \dots, I_{n,n-1}$.

The Poincaré–Birkhoff–Witt theorem for the algebra $U'_q(\mathfrak{so}_n)$ can be formulated as follows (a proof of this theorem is given in [17]): *The elements*

$$I_{21}^{+m_{21}} I_{31}^{+m_{31}} \dots I_{n1}^{+m_{n1}} I_{32}^{+m_{32}} I_{42}^{+m_{42}} \dots I_{n2}^{+m_{n2}} \dots I_{n,n-1}^{+m_{n,n-1}}, \quad m_{ij} = 0, 1, 2, \dots, \tag{8}$$

form a basis of the algebra $U'_q(\mathfrak{so}_n)$. This assertion is true if I_{ij}^+ are replaced by the corresponding elements I_{ij}^- .

Example 1. Let us consider the case of the algebra $U'_q(\mathfrak{so}_3)$. It is generated by two elements I_{21} and I_{32} , satisfying the relations

$$I_{21}^2 I_{32} - (q - q^{-1}) I_{21} I_{32} I_{21} + I_{32} I_{21}^2 = -I_{32}, \tag{9}$$

$$I_{21} I_{32}^2 - (q + q^{-1}) I_{32} I_{21} I_{32} + I_{32}^2 I_{21} = -I_{21}. \tag{10}$$

Introducing the element $I_{31}^+ \equiv I_{31} = q^{1/2} I_{21} I_{32} - q^{-1/2} I_{32} I_{21}$ we have for I_{21}, I_{32}, I_{31} the relations

$$[I_{21}, I_{32}]_q = I_{31}, \quad [I_{32}, I_{31}]_q = I_{21}, \quad [I_{31}, I_{21}]_q = I_{32}, \tag{11}$$

where the q -commutator $[\cdot, \cdot]_q$ is defined as $[A, B]_q = q^{1/2} AB - q^{-1/2} BA$.

Note that the algebra $U'_q(\mathfrak{so}_3)$ has a large automorphism group. In fact, it is seen from (9) and (10) that these relations do not change if we permute I_{21} and I_{32} . From relations (11) we see that the set of these relations do not change under cyclic permutation of the elements I_{21}, I_{32}, I_{31} . The change of a sign at I_{21} or at I_{32} also does not change the relations (9) and (10). Generating a group by these automorphisms, we may find that they generate the group isomorphic to the modular group $SL(2, \mathbb{Z})$. It is why the algebra $U'_q(\mathfrak{so}_3)$ is interesting for algebraic geometry and quantum gravity (see, for example, [18] and [19]).

Example 2. Let us consider the case of the algebra $U'_q(\mathfrak{so}_4)$. It is generated by the elements I_{21}, I_{32} and I_{43} . We create the elements

$$I_{31} = [I_{21}, I_{32}]_q, \quad I_{42} = [I_{32}, I_{43}]_q, \quad I_{41} = [I_{21}, I_{42}]_q. \tag{12}$$

Then the elements $I_{ij}, i > j$, satisfy the following set of relations

$$\begin{aligned} [I_{21}, I_{32}]_q &= I_{31}, & [I_{32}, I_{31}]_q &= I_{21}, & [I_{31}, I_{21}]_q &= I_{32}, \\ [I_{32}, I_{43}]_q &= I_{42}, & [I_{43}, I_{42}]_q &= I_{32}, & [I_{42}, I_{32}]_q &= I_{43}, \\ [I_{31}, I_{43}]_q &= I_{41}, & [I_{43}, I_{41}]_q &= I_{31}, & [I_{41}, I_{31}]_q &= I_{43}, \\ [I_{21}, I_{42}]_q &= I_{41}, & [I_{42}, I_{41}]_q &= I_{21}, & [I_{41}, I_{21}]_q &= I_{42}, \\ [I_{21}, I_{43}] &= 0, & [I_{32}, I_{41}] &= 0, & [I_{42}, I_{31}] &= (q - q^{-1})(I_{21}I_{43} - I_{32}I_{41}) \end{aligned}$$

which completely determine the algebra $U'_q(\mathfrak{so}_4)$. At $q = 1$ these relations define just the Lie algebra $\mathfrak{so}(4, \mathbb{C})$. Each of the sets $(I_{21}, I_{32}, I_{31}), (I_{32}, I_{43}, I_{42}), (I_{31}, I_{43}, I_{41}), (I_{21}, I_{42}, I_{41})$ determine a subalgebra isomorphic to $U'_q(\mathfrak{so}_3)$.

The algebra $U'_q(\mathfrak{so}_4)$ is also important for quantum gravity and algebraic geometry (see [20] and [21]). The algebra $U'_q(\mathfrak{so}_n)$ for general n is also used in quantum gravity [22].

Let us describe the automorphism group G of the algebra $U'_q(\mathfrak{so}_n)$. It is clear from the defining relations of the algebra $U'_q(\mathfrak{so}_n)$ that for each i ($i = 2, 3, \dots, n$) this algebra admits an automorphism τ_i given by the formulas

$$\tau_i : I_{j,j-1} \rightarrow I_{j,j-1}, \quad j \neq i, \quad \tau_i : I_{i,i-1} \rightarrow -I_{i,i-1}.$$

These automorphisms generate a group of automorphisms which will be denoted by G . Elements of G can be denoted by $g = (\epsilon_2, \epsilon_3, \dots, \epsilon_n)$, where ϵ_j runs independently the values $+1$ and -1 . Namely, if under action of g generating elements $I_{j_1,j_1-1}, \dots, I_{j_s,j_s-1}$ change a sign, then in $g = (\epsilon_2, \epsilon_3, \dots, \epsilon_n)$ $\epsilon_{j_1} = \dots = \epsilon_{j_s} = -1$ and other ϵ_i are equal to 1. It is clear that the group G has 2^{n-1} elements.

If $n = 3$, then the group G does not coincide with the group of all automorphisms of $U'_q(\mathfrak{so}_3)$. It is not known if this assertion is true for $n > 3$.

3 Representations of classical and nonclassical types

The elements of the set

$$I_{21}, I_{43}, \dots, I_{2k,2k-1}, \tag{13}$$

where $n = 2k$ if n is even and $n = 2k + 1$ if n is odd, commute pairwise.

Proposition 1. (a) *If T is a finite dimensional irreducible representation of the algebra $U'_q(\mathfrak{so}_n)$, then the operators*

$$T(I_{21}), T(I_{43}), \dots, T(I_{2k,2k-1})$$

are simultaneously diagonalizable.

(b) Possible eigenvalues of any of these operators can be as $i[m]_q$, $m \in \frac{1}{2}\mathbb{Z}$, or as $[m]_+$, $m \in \frac{1}{2}\mathbb{Z}$, $m \notin \mathbb{Z}$, where

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]_+ = \frac{q^m + q^{-m}}{q - q^{-1}}.$$

Eigenvalues of the form $i[m]_q$ are called *eigenvalues of the classical type*. Eigenvalues of the form $[m]_+$ are called *eigenvalues of the nonclassical type*.

The following proposition is important for construction of weight theory for finite dimensional representations of $U'_q(\mathfrak{so}_n)$.

Proposition 2. *Let T be a finite dimensional irreducible representation of $U'_q(\mathfrak{so}_n)$. Then*

(a) *Eigenvalues of any operator $T(I_{2i,2i-1})$ are all of the classical type or all of the nonclassical type.*

(b) *Moreover, all operators $T(I_{2i,2i-1})$, $i = 1, 2, \dots, k$, have eigenvalues of the same type.*

This proposition is proved by restricting the representation T to the subalgebras $U'_q(\mathfrak{so}_4)$ generated by the elements $I_{j,j-1}, I_{j+1,j}, I_{j+2,j+1}$, $j = 2, 3, \dots, n - 2$ and using the results of the paper [8].

Definition 1. A finite dimensional irreducible representation T of the algebra $U'_q(\mathfrak{so}_n)$ is called a representation of *classical (nonclassical) type* if the operators $T(I_{2i,2i-1})$, $i = 1, 2, \dots, k$ have eigenvalues of the classical (of the nonclassical) type.

Proposition 3. *Let T be a finite dimensional irreducible representation of $U'_q(\mathfrak{so}_n)$ of the classical (nonclassical) type. Then a restriction of T to the subalgebra $U'_q(\mathfrak{so}_{n-1})$ decomposes into a direct sum of irreducible representations of this subalgebra belonging to the same type.*

4 Weights of representations

In this section we construct a q -analogue of weights for finite dimensional irreducible representations of the algebra $U'_q(\mathfrak{so}_n)$. Note that this algebra has no elements which can be treated as root elements (similar to root elements of semisimple Lie algebras or quantized universal enveloping algebras of Drinfeld and Jimbo). For this reason, we do not have a weight theory for finite dimensional representations of $U'_q(\mathfrak{so}_n)$ similar to that for semisimple Lie algebras. However, we can construct the theory which can replace the weight theory of representations of semisimple Lie algebras.

Definition 2. Let T be a finite dimensional representation of the algebra $U'_q(\mathfrak{so}_n)$. Eigenvectors \mathbf{v} of operators $T(I_{2j,2j-1})$, $j = 1, 2, \dots, k$, are called *weight vectors* of the representation T . If $T(I_{2j,2j-1})\mathbf{v} = m_j\mathbf{v}$, then the set of numbers $\mathbf{m} = (m_1, m_2, \dots, m_k)$, where $n = 2k + 1$ or $n = 2k$, is called a weight of the vector \mathbf{v} .

The set of all weights of an irreducible representation T of $U'_q(\mathfrak{so}_n)$ is called a *weight diagram* of the representation T .

Proposition 4. *A weight diagram of a finite dimensional irreducible representation T of the classical type is invariant with respect to the Weyl group W of the Lie algebra $\mathfrak{so}(n, \mathbb{C})$.*

This proposition is proved by restriction of the representation T to the subalgebras $U'_q(\mathfrak{so}_3)$ generated by the pairs of generators $I_{2j,2j-1}, I_{2j+1,2j}$, $j = 1, 2, \dots$, and using the results of the paper [23].

Note that a weight diagram of a finite dimensional irreducible representation of the nonclassical type is not invariant with respect to the Weyl group W .

5 Raising and lowering operators

Recall that in the Lie algebra $\mathfrak{so}(n, \mathbb{C})$ there exist root elements $E_{\alpha_1}, E_{\alpha_2}, \dots, E_{\alpha_k}$, corresponding to simple roots, and root elements $F_{\alpha_1}, F_{\alpha_2}, \dots, F_{\alpha_k}$, corresponding to simple roots taken with sign minus. If T' is a finite dimensional irreducible representation of $\mathfrak{so}(n, \mathbb{C})$ and $|\mathbf{m}\rangle$ is its weight vector, then

$$T'(E_{\alpha_i})|\mathbf{m}\rangle = \beta_{\mathbf{m}}|\mathbf{m} + \alpha_i\rangle, \quad T'(F_{\alpha_i})|\mathbf{m}\rangle = \gamma_{\mathbf{m}}|\mathbf{m} - \alpha_i\rangle,$$

where $\beta_{\mathbf{m}}$ and $\gamma_{\mathbf{m}}$ are complex numbers. In the algebra $U'_q(\mathfrak{so}_n)$ there exist no elements similar to E_{α_j} and F_{α_j} . However, in finite dimensional representations of $U'_q(\mathfrak{so}_n)$ there exist operators having properties of the operators $T'(E_{\alpha_i})$ and $T'(F_{\alpha_i})$. These operators depend on a weight on which they act and are called *raising and lowering operators* of the representation. They are described as follows.

Let T be a finite dimensional irreducible representation of $U'_q(\mathfrak{so}_n)$ of the classical type and let $|\mathbf{m}\rangle$ be its weight vector. If $n = 2k$ we create the operators

$$\begin{aligned} R_{\alpha_i}^{\mathbf{m}} &= -T(I_{2i+2,2i-1}) + q^{-(m_i+m_{i+1})/2}T(I_{2i+1,2i}) - iq^{-m_i+1/2}T(I_{2i+2,2i}) \\ &\quad - iq^{-m_{i+1}-1/2}T(I_{2i+1,2i-1}), \quad i = 1, 2, \dots, k-1, \end{aligned} \quad (14)$$

$$\begin{aligned} L_{\alpha_i}^{\mathbf{m}} &= -T(I_{2i+2,2i-1}) + q^{(m_i+m_{i+1})/2}T(I_{2i+1,2i}) + iq^{m_i+1/2}T(I_{2i+2,2i}) \\ &\quad + iq^{m_{i+1}-1/2}T(I_{2i+1,2i-1}), \quad i = 1, 2, \dots, k-1, \end{aligned} \quad (15)$$

and the operators

$$\begin{aligned} R_{\alpha_k}^{\mathbf{m}} &= T(I_{2k,2k-3}) + q^{(-m_{k-1}+m_k)/2}T(I_{2k-1,2k-2}) + iq^{-m_{k-1}+1/2}T(I_{2k,2k-2}) \\ &\quad - iq^{m_k-1/2}T(I_{2k-1,2k-3}), \end{aligned} \quad (16)$$

$$\begin{aligned} L_{\alpha_k}^{\mathbf{m}} &= -T(I_{2k,2k-3}) + q^{(m_{k-1}-m_k)/2}T(I_{2k-1,2k-2}) + iq^{m_{k-1}+1/2}T(I_{2k,2k-2}) \\ &\quad + iq^{-m_k-1/2}T(I_{2k-1,2k-3}). \end{aligned} \quad (17)$$

If $n = 2k + 1$, then we create the operators (14), (15) and the operators

$$R_{\alpha_k}^{\mathbf{m}} = T(I_{2k+1,2k-1}) + iq^{-m_k+1/2}T(I_{2k+1,2k}), \quad (18)$$

$$L_{\alpha_k}^{\mathbf{m}} = T(I_{2k+1,2k-1}) - iq^{m_k+1/2}T(I_{2k+1,2k}). \quad (19)$$

If T is a finite dimensional representation of $U'_q(\mathfrak{so}_n)$ of the nonclassical type and $|\mathbf{m}\rangle$ is its weight vector, then we create the operators

$$\begin{aligned} R_{\alpha_i}^{\mathbf{m}} &= -T(I_{2i+2,2i-1}) + q^{-(m_i+m_{i+1})/2}T(I_{2i+1,2i}) - q^{-m_i+1/2}T(I_{2i+2,2i}) \\ &\quad - q^{-m_{i+1}-1/2}T(I_{2i+1,2i-1}), \quad i = 1, 2, \dots, k-1, \end{aligned} \quad (20)$$

$$\begin{aligned} L_{\alpha_i}^{\mathbf{m}} &= -T(I_{2i+2,2i-1}) + q^{(m_i+m_{i+1})/2}T(I_{2i+1,2i}) - q^{m_i+1/2}T(I_{2i+2,2i}) \\ &\quad - q^{m_{i+1}-1/2}T(I_{2i+1,2i-1}), \quad i = 1, 2, \dots, k-1, \end{aligned} \quad (21)$$

and the operators

$$\begin{aligned} R_{\alpha_k}^{\mathbf{m}} &= T(I_{2k,2k-3}) + q^{(-m_{k-1}+m_k)/2}T(I_{2k-1,2k-2}) + q^{-m_{k-1}+1/2}T(I_{2k,2k-2}) \\ &\quad + q^{m_k-1/2}T(I_{2k-1,2k-3}), \end{aligned} \quad (22)$$

$$\begin{aligned} L_{\alpha_k}^{\mathbf{m}} &= -T(I_{2k,2k-3}) - q^{(m_{k-1}-m_k)/2}T(I_{2k-1,2k-2}) + q^{m_{k-1}+1/2}T(I_{2k,2k-2}) \\ &\quad + q^{-m_k-1/2}T(I_{2k-1,2k-3}) \end{aligned} \quad (23)$$

if $n = 2k$. If $n = 2k + 1$, then we create the operators (20), (21) and the operators

$$R_{\alpha_k}^{\mathbf{m}} = T(I_{2k+1,2k-1}) + q^{-m_k+1/2}T(I_{2k+1,2k}), \tag{24}$$

$$L_{\alpha_k}^{\mathbf{m}} = T(I_{2k+1,2k-1}) - q^{m_k+1/2}T(I_{2k+1,2k}). \tag{25}$$

The operators $R_{\alpha_k}^{\mathbf{m}}$ and $L_{\alpha_k}^{\mathbf{m}}$ correspond to the operators $T'(E_{\alpha_i})$ and $T'(F_{\alpha_i})$ of a representation T' of the Lie algebra $\mathfrak{so}(n, \mathbb{C})$, respectively. We have

$$R_{\alpha_k}^{\mathbf{m}}|\mathbf{m}\rangle = \beta_i|\mathbf{m} + \alpha_i\rangle, \quad L_{\alpha_k}^{\mathbf{m}}|\mathbf{m}\rangle = \gamma_i|\mathbf{m} - \alpha_i\rangle, \tag{26}$$

where α_i and γ_i are complex numbers, which depend on the representation of $U'_q(\mathfrak{so}_n)$. Note that the relations (26) are not true if we replace the vector $|\mathbf{m}\rangle$ by some other weight vector $|\mathbf{m}'\rangle$, since in such a case in the right hand side we shall obtain, beside the vectors $|\mathbf{m}' + \alpha_i\rangle$ and $|\mathbf{m}' - \alpha_i\rangle$, other weight vectors.

Formulas (14)–(17) and (20)–(23) for raising and lowering operators follow from formulas of section 8 of the paper [8] if to restrict the representation T of $U'_q(\mathfrak{so}_n)$ to the subalgebras $U'_q(\mathfrak{so}_4)$ generated by the elements $I_{2j,2j-1}, I_{2j+1,2j}, I_{2j+2,2j+1}, j = 1, 2, \dots, k - 1$.

Formulas (18), (19), (24) and (25) for raising and lowering operators follow from formulas for raising and lowering operators for irreducible representations of the algebra $U'_q(\mathfrak{so}_3)$ of the paper [8] if to restrict the representation T to the subalgebra $U'_q(\mathfrak{so}_3)$ generated by the elements $I_{2k+1,2k}$ and $I_{2k,2k-1}$.

Definition 3. If T is a finite dimensional irreducible representation of the algebra $U'_q(\mathfrak{so}_n)$, then a weight \mathbf{m} of this representation is called a *highest weight* if $R_{\alpha_i}^{\mathbf{m}}|\mathbf{m}\rangle = 0, i = 1, 2, \dots, k$. The corresponding vector $|\mathbf{m}\rangle$ is called a *highest weight vector*.

Let us give a form of highest weights of irreducible representations of the classical and of the nonclassical types. In order to determine such a form we restrict the corresponding irreducible representations of $U'_q(\mathfrak{so}_n)$ to the subalgebras $U'_q(\mathfrak{so}_4)$ and $U'_q(\mathfrak{so}_3)$ and use the results of the papers [8] and [23]. As a result, we find that if a weight $\mathbf{m} \equiv (m_1, m_2, \dots, m_k)$ of an irreducible representation T of the classical type is a highest weight, then the numbers m_j are all integral or all half-integral (but not integral) and satisfy the conditions

$$m_1 \geq m_2 \geq \dots \geq m_k \quad \text{if } n = 2k + 1$$

and the conditions

$$m_1 \geq m_2 \geq \dots \geq m_{k-1} \geq |m_k| \quad \text{if } n = 2k.$$

The set of these highest weights coincides with the set of highest weights of irreducible finite dimensional representations of the Lie algebra $\mathfrak{so}(n, \mathbb{C})$. These highest weights will be called *highest weights of the classical type*.

If a weight $\mathbf{m} \equiv (m_1, m_2, \dots, m_k)$ of an irreducible representation T of the nonclassical type is a highest weight, then the numbers m_j are all half-integral (but not integral). In order to formulate the classification theorem for representations of the nonclassical type we shall need only highest weights \mathbf{m} for which all m_j are positive. Such highest weights must satisfy the conditions

$$m_1 \geq m_2 \geq \dots \geq m_k \geq 1/2.$$

These highest weights will be called *highest weights of the nonclassical type*.

It is well known that the root elements E_{α_i} and F_{α_i} of the Lie algebra $\mathfrak{so}(n, \mathbb{C})$ satisfy the relations

$$[E_{\alpha_i}, F_{\alpha_i}] = 2H_{\alpha_i}, \quad [E_{\alpha_i}, F_{\alpha_j}] = 0, \quad i \neq j.$$

Instead of these relations for raising and lowering operators of representations of the classical and nonclassical type of the algebra $U'_q(\mathfrak{so}_{2k})$ we have the relations

$$(R_{\alpha_i}^{\mathbf{m}-\alpha_i} L_{\alpha_i}^{\mathbf{m}} - L_{\alpha_i}^{\mathbf{m}+\alpha_i} R_{\alpha_i}^{\mathbf{m}}) |\mathbf{m}\rangle = [2l]_q \left\{ (q - q^{-1})^2 C_4 - (q^{2l} + q^{-2l}) (q^2 - q^{-2}) \right\} |\mathbf{m}\rangle, \quad (27)$$

$$(R_{\alpha_i}^{\mathbf{m}-\alpha_j} L_{\alpha_j}^{\mathbf{m}} - L_{\alpha_j}^{\mathbf{m}+\alpha_i} R_{\alpha_i}^{\mathbf{m}}) |\mathbf{m}\rangle = 0, \quad (28)$$

where $l = (m_i - m_{i+1})/2$ if $i \neq k$ and $l = (m_i + m_{i+1})/2$ if $i = k$ and C_4 is the Casimir operator of the subalgebra $U'_q(\mathfrak{so}_4)$ generated by the elements $I_{2i,2i-1}, I_{2i+1,2i}, I_{2i+2,2i+1}$, which is given as

$$C_4 = q^{-1} I_{2i,2i-1} I_{2i+2,2i+1} - I_{2i+1,2i-1} I_{2i+2,2i} + q I_{2i+1,2i} I_{2i+2,2i-1}.$$

For the algebra $U'_q(\mathfrak{so}_{2k+1})$ we have the relations (27) for $i \neq k$, (28) for $i \neq j$ and the relations

$$(R_{\alpha_k}^{\mathbf{m}-\alpha_k} L_{\alpha_k}^{\mathbf{m}} - L_{\alpha_k}^{\mathbf{m}+\alpha_k} R_{\alpha_k}^{\mathbf{m}}) |\mathbf{m}\rangle = q[m_k]_q [m_k]_+ (q - q^{-1})^2 |\mathbf{m}\rangle. \quad (29)$$

6 Classification theorems

For finite dimensional irreducible representations of the classical type the following theorem is true.

Theorem 2. (a) *Each irreducible finite dimensional representation of the classical type has a highest weight. A highest weight is unique (up to a constant).*

(b) *Irreducible finite dimensional representations with different highest weights are not equivalent. Conversely, nonequivalent irreducible finite dimensional representations of $U'_q(\mathfrak{so}_n)$ have different highest weights.*

Existing of a highest weight is proved in the same way as in the case of irreducible representations of the Lie algebra $\mathfrak{so}(n, \mathbb{C})$ by using Propositions 1 and 2. A proof of uniqueness of highest weight is not simple. The relations (27)–(29) are used in this proof.

The assertion (b) is proved by using a proof of the similar assertion for irreducible representations of the algebra $U'_q(\mathfrak{so}_4)$ from paper [8]. Namely, if $|\mathbf{m}\rangle$ is a highest weight vector, then we act upon $|\mathbf{m}\rangle$ successively by the corresponding operators $L_{\alpha_i}^{\mathbf{m}'}$, $i = 1, 2, \dots, k$. Then, as in [8], we can find how the operators $R_{\alpha_i}^{\mathbf{m}'}$ act upon weight vectors $|\mathbf{m}'\rangle$. Therefore, by the method of the paper [8] we evaluate uniquely how the operators $T(I_{2i+2,2i-1}), T(I_{2i+1,2i}), T(I_{2i+2,2i}), T(I_{2i+2,2i+1})$ act upon the corresponding weight vectors. Thus, a highest weight determines uniquely (up to equivalence) the operators $T(I_{j,j-1})$, $j = 2, 3, \dots, n$.

Thus, in order to obtain a classification of irreducible finite dimensional representations of the classical type of the algebra $U'_q(\mathfrak{so}_n)$ we have to determine highest weights, described in the previous section, to which such irreducible representations with these highest weights correspond.

It can be proved that the irreducible representation $T_{\mathbf{m}}$ of $U'_q(\mathfrak{so}_n)$ from the paper [5] are of the classical type and has highest weight \mathbf{m} . If we take all these irreducible representations $T_{\mathbf{m}}$, then they give all highest weights \mathbf{m} , described in previous section for irreducible representations of the classical type. That is, for each highest weight \mathbf{m} of the classical type from the previous section there corresponds an irreducible representation of $U'_q(\mathfrak{so}_n)$. Thus, we obtain the following classification of irreducible representations of the classical type.

Theorem 3. *Irreducible finite dimensional representations of the classical type of the algebra $U'_q(\mathfrak{so}_n)$ are in one-to-one correspondence with highest weights of the classical type, described in the previous section.*

Thus, irreducible finite dimensional representations of the classical type of the algebra $U'_q(\mathfrak{so}_n)$ are in one-to-one correspondence with irreducible finite dimensional representations of the Lie algebra \mathfrak{so}_n . The corresponding irreducible representations of $U'_q(\mathfrak{so}_n)$ and of \mathfrak{so}_n act on the same vector space. Moreover, when $q \rightarrow 1$, then operators of an irreducible representation of $U'_q(\mathfrak{so}_n)$ tend to the corresponding operators of the corresponding irreducible representation of \mathfrak{so}_n . This is a reason why the representations of Theorem 3 are called representations of the classical type.

An analogue of Theorem 2 for irreducible representations of the nonclassical type is formulated as follows.

Theorem 4. (a) *Each irreducible finite dimensional representation of the nonclassical type has a highest weight. A highest weight is unique (up to a constant).*

(b) *Irreducible finite dimensional representations with different highest weights are not equivalent.*

This theorem is proved in the same way as Theorem 2.

In order to formulate the classification theorem for irreducible representations of the nonclassical type we first formulate the following proposition.

Proposition 5. *If T is an irreducible representation of the nonclassical type and G is the automorphism group of $U'_q(\mathfrak{so}_n)$ from section 2, then the composition $T^{(g)} := T \circ g$, $g \in G$, $g \neq e$, is a representation of the nonclassical type which is not equivalent to T .*

This proposition is proved by showing that spectrum of the operator $T(I_{2i,2i-1})$ ($i = 1, 2, \dots, k$) coincides with the set $[\frac{1}{2}]_+, [\frac{3}{2}]_+, \dots, [\frac{s}{2}]_+$ or with the set $-\frac{1}{2}_+, -\frac{3}{2}_+, \dots, -\frac{s}{2}_+$, where s is some positive integer. In order to show this we use the method of mathematical induction. For $U'_q(\mathfrak{so}_4)$ this assertion is true (see [8]). The induction is proved by using Wigner–Eckart theorem for irreducible representations of the nonclassical type derived by N. Iorgov (this theorem will be published).

Thus, with every irreducible representation T of the nonclassical type we associate a set of irreducible representations $\{T^{(g)} \mid g \in G\}$, consisting of 2^{n-1} pairwise nonequivalent irreducible representations of the nonclassical type. In this set there exists exactly one irreducible representation with highest weight $\mathbf{m} = (m_1, m_2, \dots, m_k)$ such that $m_1 \geq m_2 \geq \dots \geq m_k \geq \frac{1}{2}$.

For every highest weight of the nonclassical type \mathbf{m} with $m_1 \geq m_2 \geq \dots \geq m_k \geq \frac{1}{2}$ there exists an irreducible representation of the nonclassical type having \mathbf{m} as its highest weight. Therefore, from above reasoning we derive the following classification of irreducible representations of the nonclassical type.

Theorem 5. *Irreducible representations of the nonclassical type of the algebra $U'_q(\mathfrak{so}_n)$ are in one-to-one correspondence with pairs (\mathbf{m}, g) , where \mathbf{m} is a highest weight of the nonclassical type with $m_1 \geq m_2 \geq \dots \geq m_k \geq \frac{1}{2}$ and g is an element of the automorphism group G .*

Note that irreducible representations of the nonclassical type have no classical analogue. Namely, operators of representations of the nonclassical type are singular at the point $q = 1$.

7 Irreducible representations of $U'_q(\mathfrak{so}_3)$

This and the next sections are devoted to examples of the theory described above. In this section we describe irreducible finite dimensional representations of the algebra $U'_q(\mathfrak{so}_3)$.

Irreducible finite dimensional representations of the classical type of this algebra are given by nonnegative integral or half-integral number l . The irreducible representation T_l , given by such

a number l , acts on $(2l + 1)$ -dimensional vector space \mathcal{H}_l with a basis $|l, m\rangle$, $m = -l, -l + 1, \dots, l$. The operators $T_l(I_{21})$ and $T_l(I_{32})$ are given by the formulas

$$T_l(I_{21})|l, m\rangle = i[m]_q|l, m\rangle,$$

$$T_l(I_{32})|l, m\rangle = \frac{1}{q^m + q^{-m}} ([l - m]_q|l, m + 1\rangle - [l + m]_q|l, m - 1\rangle),$$

where $[a]_q$ denotes a q -number. Note that for these representations we have

$$\text{Tr } T_l(I_{21}) = 0, \quad \text{Tr } T_l(I_{32}) = 0.$$

Irreducible representations $T_n^{\epsilon_1, \epsilon_2}$ of the nonclassical type are given by the numbers $\epsilon_i = \pm 1$ (they determine elements of the automorphism group G) and by the integer $n = 1, 2, \dots$ (According to Section 6, these representations are given by half-integral number l , but we replaced l by $n = l + 1/2$.) The representation $T_n^{\epsilon_1, \epsilon_2}$ acts on n -dimensional vector space with the basis $|k\rangle$, $k = 1, 2, \dots, n$. The operators $T_n^{\epsilon_1, \epsilon_2}(I_{21})$ and $T_n^{\epsilon_1, \epsilon_2}(I_{32})$ are given by the formulas

$$T_n^{\epsilon_1, \epsilon_2}(I_{21})|k\rangle = \epsilon_1 \frac{q^{k-1/2} + q^{-k+1/2}}{q - q^{-1}} |k\rangle,$$

$$T_n^{\epsilon_1, \epsilon_2}(I_{32})|1\rangle = \frac{1}{q^{1/2} - q^{-1/2}} (\epsilon_2[n]_q|1\rangle + i[n - 1]_q|2\rangle),$$

$$T_n^{\epsilon_1, \epsilon_2}(I_{32})|k\rangle = \frac{1}{q^{k-1/2} - q^{-k+1/2}} (i[n - k]_q|k + 1\rangle) + i[n + k - 1]_q|k - 1\rangle),$$

These representations have the properties

$$\text{Tr } T_n^{\epsilon_1, \epsilon_2}(I_{21}) \neq 0, \quad \text{Tr } T_n^{\epsilon_1, \epsilon_2}(I_{32}) \neq 0.$$

There exist 4 one-dimensional irreducible representations of the nonclassical type. They are equivalent to $T_1^{\epsilon_1, \epsilon_2}$, $\epsilon_i = \pm 1$.

Note that a proof of the fact that these representations of $U'_q(\mathfrak{so}_3)$ exhaust all irreducible representations of this algebra is given in [23].

8 Irreducible representations of $U'_q(\mathfrak{so}_4)$

Irreducible finite dimensional representations of the classical type of the algebra $U'_q(\mathfrak{so}_4)$ are given by two integral or two half-integral (but not integral) numbers r and s such that $r \geq |s|$. These numbers constitute the highest weight of the representation. We define the numbers $j = (r + s)/2$ and $j' = (r - s)/2$ and denote the representation by $T_{jj'}$. This representation acts on the vector space with the basis

$$|k, l\rangle, \quad k = -j, -j + 1, \dots, j, \quad l = -j', -j' + 1, \dots, j'.$$

The operators $T_{jj'}(I_{i, i-1})$, $i = 2, 3, 4$, act upon these vectors by the formulas

$$T_{jj'}(I_{21})|k, l\rangle = i[k + l]_q|k, l\rangle, \quad T_{jj'}(I_{43})|k, l\rangle = i[k - l]_q|k, l\rangle,$$

$$T_{jj'}(I_{32})|k, l\rangle = \frac{1}{(q^{k+l} + q^{-k-l})(q^{k-l} + q^{-k+l})}$$

$$\times \left\{ - \left(q^{j-l} + q^{-j+l} \right) [j' - l]_q|k, l + 1\rangle + \left(q^{j+l} + q^{-j-l} \right) [j' + l]_q|k, l - 1\rangle \right.$$

$$\left. + \left(q^{j'-k} + q^{-j'+k} \right) [j - k]_q|k + 1, l\rangle - \left(q^{j'+k} + q^{-j'-k} \right) [j + k]_q|k - 1, l\rangle \right\}.$$

Irreducible finite dimensional representations of the nonclassical type of the algebra $U'_q(\mathfrak{so}_4)$ are given by two half-integral (but not integral) numbers r, s such that $r \geq s > 0$ and by the numbers $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_i = \pm 1$, which determine elements of the automorphism group G . The numbers r and s constitute a highest weight of the representation if $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$. We define the numbers $j = (r + s)/2$ and $j' = (r - s)/2$ and denote the corresponding representations by $T_{jj'}^{\epsilon_1, \epsilon_2, \epsilon_3}$.

If (r, s) runs over all highest weights of the nonclassical type with $r \geq s > 0$, then j and j' run over the values

$$j = 0, 1, 2, \dots, \quad j' = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \quad \text{or} \quad j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots, \quad j' = 0, 1, 2, \dots$$

The representation $T_{jj'}^{\epsilon_1, \epsilon_2, \epsilon_3}$ acts on the vector space \mathcal{H} with the basis

$$|k, l\rangle, \quad k = j, j - 1, \dots, \frac{1}{2}, \quad l = j', j' - 1, \dots, -j',$$

if j' is integral and with the basis

$$|k, l\rangle, \quad k = j, j - 1, \dots, -j, \quad l = j', j' - 1, \dots, \frac{1}{2},$$

if j is integral. The representations are given by the formulas

$$\begin{aligned} T_{jj'}^{\epsilon_1, \epsilon_2, \epsilon_3}(I_{21})|k, l\rangle &= \epsilon_1[k + l]_+ |k, l\rangle, & T_{jj'}^{\epsilon_1, \epsilon_2, \epsilon_3}(I_{43})|k, l\rangle &= \epsilon_2[k - l]_+ |k, l\rangle, \\ T_{jj'}^{\epsilon_1, \epsilon_2, \epsilon_3}(I_{32})|k, l\rangle &= \frac{1}{[k + l]_q [k - l]_q (q - q^{-1})} \{ -i[j' - l]_q [j - l]_q |k, l + 1\rangle \\ &+ i[j' + l]_q [j + l]_q |k, l - 1\rangle - i[j' - k]_q [j - k]_q |k + 1, l\rangle + i[j' + k]_q [j + k]_q |k - 1, l\rangle \}, \end{aligned}$$

where $k \neq \frac{1}{2}$ if j is half-integral and $l \neq \frac{1}{2}$ if j' is half-integral, and by

$$\begin{aligned} T_{jj'}^{\epsilon_1, \epsilon_2, \epsilon_3}(I_{32})|\frac{1}{2}, l\rangle &= \frac{1}{[l + \frac{1}{2}]_q [l - \frac{1}{2}]_q (q - q^{-1})} \{ -i[j - l]_q [j' - l]_q |\frac{1}{2}, l + 1\rangle \\ &+ i[j + l]_q [j' + l]_q |\frac{1}{2}, l - 1\rangle - i[j' - \frac{1}{2}]_q [j - \frac{1}{2}]_q |\frac{3}{2}, l\rangle + i[j' + \frac{1}{2}]_q [j + \frac{1}{2}]_q \epsilon_3 (-1)^l |\frac{1}{2}, -l\rangle \} \end{aligned}$$

if j is half-integral and by

$$\begin{aligned} T_{jj'}^{\epsilon_1, \epsilon_2, \epsilon_3}(I_{32})|k, \frac{1}{2}\rangle &= \frac{1}{[k + \frac{1}{2}]_q [k - \frac{1}{2}]_q (q - q^{-1})} \{ -i[j - \frac{1}{2}]_q [j' - \frac{1}{2}]_q |k, \frac{3}{2}\rangle + i[j + \frac{1}{2}]_q [j' + \frac{1}{2}]_q \\ &\times \epsilon_3 (-1)^k | -k, \frac{1}{2}\rangle - i[j' - k]_q [j - k]_q |k + 1, \frac{1}{2}\rangle + i[j' + k]_q [j + k]_q + q |k - 1, \frac{1}{2}\rangle \} \end{aligned}$$

if j' is half-integral.

Note that a proof of the fact that these representations of $U'_q(\mathfrak{so}_4)$ exhaust all irreducible representations of this algebra is given in [8].

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