Linear Theory of 6D Ionization Cooling

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Ionization cooling is the best-known beam cooling mechanism for the envisioned muon colliders and neutrino factories. Although the basic mechanism of ionization cooling is very similar to the well-established radiation damping of electron beam in storage rings, many challenges in beam physics as well as accelerator engineering have to be met in order to design and implement an ionization cooling channel. In this paper we describe the beam physics aspect of ionization cooling and review our current understanding of linear beam dynamics, especially the various heating mechanisms involved in the cooling process. Our focus is on the linear theory of 6D ionization cooling in solenoidal channels. Status of cooling channel developments is briefly addressed. This paper is intended to be readable for nonspecialists.

1. Introduction

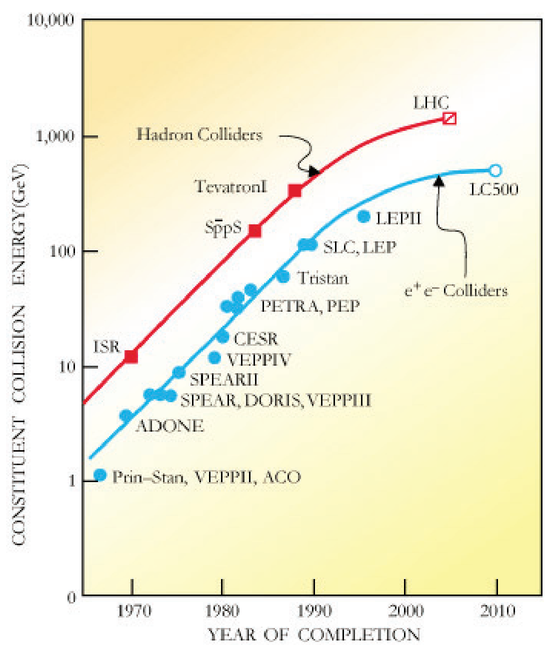

Figure 1: Effective constituent collision energy of hadron and electron-positron colliders plotted against completion date. (©AIP, Physics Today [1].

Electron and proton accelerators have been leading the high energy frontier for decades. As the energy frontier is being pushed further and further, both electron and proton colliders seem to be approaching their limits due to physical, technological, and economic constraints. This is evidenced by the falloff of the Livingston curve plotted in Fig. 1 [1, 2]. Historically the exponential growth enjoyed by the high energy physics community has been powered by innovations in accelerating technologies (especially new types of accelerators) as shown in a Livingston plot that

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indicates accelerator types [3]. When a given accelerating technology is pushed close to its limits, the growth will saturate and new technology has to take over. Within the last decade, muon colliders have been pursued as a potential contender for the energy frontier in the future. This is mainly because muons can be accelerated efficiently to very high energies in ring-like accelerators. Due to the fact that muons are 207 times heavier than electrons, muon machines do not suffer from synchrotron radiation energy loss as electron machines do. Being leptonic, muon machines can yield the same energy reach with a beam energy many times lower than that required for proton machines [2, 4]. In recent years, the possibility of using a muon storage ring as an intense neutrino source has attracted worldwide attention, and the feasibility of muon-based neutrino factories is being investigated [5, 6, 7].

The biggest challenge for high energy muon accelerators is to reduce the 6D phase-space volume of a muon beam by orders of magnitude within a fraction of the muons' decay time. Ionization cooling appears to be the only promising mechanism to provide a sufficiently short damping time. Ionization cooling is achieved by reducing the muons' momenta through ionization energy loss in absorbers and replenishing the momentum loss only in the longitudinal direction through rf cavities. This mechanism can effectively reduce the transverse phase space of a muon beam in the same way as radiation damping does to an electron beam. However this mechanism does not effectively cool the longitudinal momentum spread because the energy-loss rate is not sensitive to beam momentum except for very low-energy muons. To obtain longitudinal cooling, dispersion must be introduced to spatially separate muons of different longitudinal momenta, and wedged absorbers are used to reduce the momentum spread. Such a longitudinal cooling scheme is called "emittance exchange" because the longitudinal cooling is achieved at the expense of transverse heating or a reduced transverse cooling rate.

Besides the desired ionization energy loss, other undesirable processes take place during muons' interactions with materials. Two major effects are considered here. One is the multiple scattering of muons from the nucleus of the materials, and the other is the energy straggling resulting from the fluctuation of ionization energy loss. These effects are stochastic in nature and thus tend to diffuse the beam phase-space distribution, i.e., heating the beam. For later use, we introduce quantities η, χ, and χ_δ to quantify the ionization energy loss, multiple scattering, and energy straggling, respectively. Let p and v be the muon momentum and velocity,

$$\eta = \frac{1}{pv}\left|\frac{dE}{ds}\right| \tag{1}$$

is a positive quantity characterizing the cooling force from energy loss.

$$\chi = \left(\frac{13.6\,\mathrm{MeV}}{pv}\right)^2 \frac{1}{L_{\mathrm{rad}}} \tag{2}$$

is the mean square of the projected angular deviations due to multiple scattering, where L_{rad} is the radiation length.

$$\chi_\delta = 4\pi (r_e m_e c^2)^2 n_e \gamma^2 \left(1 - \frac{\beta}{2}\right) \simeq 0.157\,\rho \frac{Z}{A}\gamma^2\left(1-\frac{\beta}{2}\right) \qquad (\mathrm{MeV})^2\mathrm{cm}^2/\mathrm{gm} \tag{3}$$

is the mean square of the energy fluctuations due to energy straggling [8, 9].

To apply the simple concept of ionization cooling to a muon beam, much hard work must be done to limit the beam heating and reach a design that is physically sound and feasible from an engineering standpoint. Our concern here is the physical design, for which an understanding of the beam dynamics is essential. Conceptually, there are usually three different types of dynamics problems: 1) Linear dynamics that should dominate the beam behavior for a promising design. On this subject, many theoretical treatments and design experiences can be adopted from electron machines. However, unique features of muon cooling channels call for new developments. Obviously cooling and heating mechanisms are new subjects. Another not-so-new subject is solenoidal focusing. Because of the large beam size and relatively low beam energy, lattices consisting of large-aperture solenoids appear suitable for transporting the muon beam. Solenoidal channels have long been used and treated for low-energy linacs, but not to the extent needed for muon cooling channels. The theory discussed below reflects these new developments. At early design stage, linear theory is very useful because it provides the necessary understanding to reliably and quickly set the design requirements and main features of a cooling channel.

2) Nonlinear problems due to the nonlinear magnetic and rf focusing forces. Nonlinearities are responsible for the non-decay loss of muons in a channel. For muon cooling channels, the nonlinearity in both transverse and longitudinal degrees of freedom are quite severe. Careful studies are necessary, for which mature analytical and numerical tools developed for electron and proton machines can be used. 3) Collective effects due to particles in a bunch interacting with each other through direct Coulomb forces and wakefields. Because cooling channels have very large apertures and relatively low beam intensities, collective effects are not much of a concern for the time being.

In the next section, we will address the linear beam dynamics in a bare muon transport channel without absorbers. In this case, beam motion can be described as a Hamiltonian flow in the phase space. Emphasis will be on the basic beam concepts like beam emittances and envelope functions, which are essential to characterize a beam. After this, we will discuss how the perturbations from muons' interactions with materials affect the Hamiltonian flow and result in phase-space cooling and heating. Emittance evolution equations governing the cooling process will be established for solenoidal channels. Finally we will address the current status of cooling channel developments.

2. Hamiltonian flow and beam emittances

The dominating forces on muons in a cooling channel are the electromagnetic forces from the magnets and rf cavities that are designed to keep the beam well confined in phase space through a cooling channel. Interactions with materials can be treated as perturbations. A muon's motion can be conveniently described by its spatial coordinates (x, y, z) and their corresponding canonical momenta (p_x, p_y, p_z). Usually, these phase-space variables are defined relative to a reference particle whose trajectory follows the beamline layout. x and y represent deviations perpendicular to the reference trajectory and z represents the longitudinal deviation. Path length s along the reference orbit is used as the time variable, and $\delta = (p - p_0)/p_0$ is the relative longitudinal momentum deviation from the reference momentum p_0. Without materials, a muon's motion is governed by a Hamiltonian $H(x, p_x, y, p_y, z, \delta, s)$. By construct, the phase-space variables represent deviations from the reference trajectory and should be small. Thus we can expand the Hamiltonian into a polynomial series of these variables. The first-order terms can be eliminated by proper choice of the reference orbit. The second-order terms specify the linear forces that we are interested in. The third- and higher-order terms correspond to the nonlinear forces to be designed small.

Now let us consider solenoidal focusing channels. The main transverse focusing is provided by strong solenoids of longitudinal field strength B_s. To generate dispersion for emittance exchange, vertical dipole field B_y is superimposed upon the solenoidal field. Since the main solenoid field continuously rotates the beam and tends to make the beam rotationally symmetric, it is advantageous to keep symmetric focusing even with the dipole field. To achieve this, a quadrupole field is added to balance the horizontal focusing from the dipoles. The linear Hamiltonian for such a system reads [10]

$$\begin{aligned} H &= \underbrace{\frac{1}{2}\left(p_x^2 + p_y^2\right)}_{\text{drift}} + \underbrace{\frac{1}{2}\hat{b}_s^2\left(x^2 + y^2\right) - \hat{b}_s\left(xp_y - yp_x\right)}_{\text{solenoid}} \\ &\quad \underbrace{- \frac{x\delta}{\rho(s)} + \frac{x^2}{2\rho(s)^2}}_{\text{dipole}} + \underbrace{\frac{1}{2}\hat{b}_1\left(x^2 - y^2\right)}_{\text{quadrupole}} + \underbrace{\frac{1}{2}\left[\delta^2 + V(s)z^2\right]}_{\text{longitudinal and rf}}, \end{aligned} \tag{4}$$

where the normalized solenoid strength $\hat{b}_s = \frac{q}{2p_0}B_s(0,0,s)$, the curvature of the reference orbit $\frac{1}{\rho} = \frac{q}{p_0}B_y(0,0,s)$, and the normalized quadrupole strength $\hat{b}_1 = \frac{q}{p_0}\left.\frac{\partial B_y}{\partial x}\right|_{x=y=0}$. The longitudinal focusing due to rf is given by $V(s)$. For symmetric focusing, the quadrupole components must be tied to the bending radius as $\hat{b}_1(s) = -1/2\rho(s)^2$ to form gradient dipoles with field index n=1/2. Then the total focusing strength becomes $K(s) = \hat{b}_s(s)^2 + 1/2\rho(s)^2$. The Hamiltonian becomes

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}K(s)\left(x^2 + y^2\right) - \hat{b}_s(s)\left(xp_y - yp_x\right) - \frac{x\delta}{\rho(s)} + \frac{1}{2}\left[\delta^2 + V(s)z^2\right]. \tag{5}$$

We are mainly interested in cooling channels consisting of periodic cooling cells, i.e., the Hamiltonian and field strength functions are periodic in s.

To understand the beam evolution, it is critical to find all quadratic invariants of this linear Hamiltonian because they determine the shape of the equilibrium phase-space distributions. To identify the quadratic invariants, the lab-frame Hamiltonian is decoupled by two canonical transformations. The first transformation decouples the two transverse degrees of freedom by rotating to the Larmor frame, which rotates at half of the cyclotron frequency, to remove the angular momentum term. Using the ˜ over a symbol to indicate that it is in the Larmor frame, the transformation reads

$$\begin{aligned} x &= \tilde{x}\cos\theta + \tilde{y}\sin\theta \quad , \quad & p_x &= \tilde{p}_x\cos\theta + \tilde{p}_y\sin\theta \\ y &= -\tilde{x}\sin\theta + \tilde{y}\cos\theta \quad , \quad & p_y &= -\tilde{p}_x\sin\theta + \tilde{p}_y\cos\theta, \end{aligned} \tag{6}$$

where $\theta(s) = \int_0^s \hat{b}_s(\bar{s})d\bar{s}$ is the rotating angle of the Larmor frame. The second transformation decouples the transverse and longitudinal degrees of freedom via

$$\begin{aligned} &\tilde{x} = \tilde{x}_\beta + \tilde{D}_x\delta \ , \quad \tilde{y} = \tilde{y}_\beta + \tilde{D}_y\delta \ , \quad z = \hat{z} - \tilde{D}'_x\tilde{x} + \tilde{D}_x\tilde{p}_x - \tilde{D}'_y\tilde{y} + \tilde{D}_y\tilde{p}_y \\ &\tilde{p}_x = \tilde{p}_{x_\beta} + \tilde{D}'_x\delta \ , \quad \tilde{p}_y = \tilde{p}_{y_\beta} + \tilde{D}'_y\delta \ , \quad \delta = \hat{\delta}, \end{aligned} \tag{7}$$

where the dispersion functions can be determined by [10]

$$\tilde{D}''_x + K(s)\tilde{D}_x = \frac{\cos[\theta(s)]}{\rho(s)} \ , \quad \tilde{D}''_y + K(s)\tilde{D}_y = \frac{\sin[\theta(s)]}{\rho(s)}. \tag{8}$$

The resulting Hamiltonian reduces to the simple form

$$\tilde{H}_\beta = \frac{1}{2}\left(\tilde{p}^2_{x_\beta} + \tilde{p}^2_{y_\beta}\right) + \frac{1}{2}K(s)\left(\tilde{x}^2_\beta + \tilde{y}^2_\beta\right) + \frac{1}{2}\Big[I(s)\delta^2 + \mathrm{V}(s)\hat{z}^2\Big]. \tag{9}$$

Here $I(s) = 1 - \frac{\tilde{D}_x\cos[\theta(s)]}{\rho(s)} - \frac{\tilde{D}_y\sin[\theta(s)]}{\rho(s)}$ reflects the momentum compaction effect.

From the decoupled Hamiltonian, it is not difficult to confirm (find) the following five linearly independent quadratic invariants, three Courant-Snyder type of invariants for each degree of freedom and two extra invariants due to the degeneracy in the transverse degrees of freedom,

$$I_x = \gamma_T\,\tilde{x}^2_\beta + 2\alpha_T\,\tilde{x}_\beta\tilde{p}_{x_\beta} + \beta_T\,\tilde{p}^2_{x_\beta}, \tag{10}$$

$$I_y = \gamma_T\,\tilde{y}^2_\beta + 2\alpha_T\,\tilde{y}_\beta\tilde{p}_{y_\beta} + \beta_T\,\tilde{p}^2_{y_\beta}, \tag{11}$$

$$I_z = \gamma_L\,\hat{z}^2 + 2\alpha_L\,\hat{z}\delta + \beta_L\,\delta^2, \tag{12}$$

$$I_{xy} = \gamma_T\,\tilde{x}_\beta\tilde{y}_\beta + 2\alpha_T\frac{\tilde{x}_\beta\tilde{p}_{y_\beta} + \tilde{y}_\beta\tilde{p}_{x_\beta}}{2} + \beta_T\,\tilde{p}_{x_\beta}\tilde{p}_{y_\beta}, \tag{13}$$

$$L_z = \tilde{x}_\beta\tilde{p}_{y_\beta} - \tilde{y}_\beta\tilde{p}_{x_\beta}. \tag{14}$$

Here the transverse and longitudinal Courant-Snyder parameters can be determined by

$$\beta'_T = -2\alpha_T\ ,\ \alpha'_T = K(s)\beta_T - \gamma_T\ ,\ \gamma_T = \frac{1+\alpha_T^2}{\beta_T} \tag{15}$$

and

$$\beta'_L = -2I(s)\alpha_T\ ,\ \alpha'_L = V(s)\beta_T - I(s)\gamma_T\ ,\ \gamma_L = \frac{1+\alpha_L^2}{\beta_L}. \tag{16}$$

Averaged over the phase space, these five quadratic invariants lead to five beam invariants that determine the areas a beam occupies in the closed subspaces, which are preserved by the Hamiltonian flow in phase space. These invariant beam quantities are normally called beam emittances, which are important because they are conserved beam properties and characterize the density of beam phase-space distribution. In a solenoid channel, the emittances are

$$
\begin{aligned}
\varepsilon_x &\equiv \sqrt{\langle \tilde{x}_\beta^2\rangle\langle \tilde{p}_{x_\beta}^2\rangle - \langle \tilde{x}_\beta \tilde{p}_{x_\beta}\rangle^2} = \frac{1}{2}\langle I_x\rangle, && (17)\\
\varepsilon_y &\equiv \sqrt{\langle \tilde{y}_\beta^2\rangle\langle \tilde{p}_{y_\beta}^2\rangle - \langle \tilde{y}_\beta \tilde{p}_{y_\beta}\rangle^2} = \frac{1}{2}\langle I_y\rangle, && (18)\\
\varepsilon_z &\equiv \sqrt{\langle \hat{z}^2\rangle\langle \delta^2\rangle - \langle \hat{z}\delta\rangle^2} = \frac{1}{2}\langle I_z\rangle, && (19)\\
\varepsilon_{xy} &\equiv \sqrt{\langle \tilde{x}_\beta \tilde{y}_\beta\rangle\langle \tilde{p}_{x_\beta}\tilde{p}_{y_\beta}\rangle - \langle \frac{\tilde{x}_\beta \tilde{p}_{y_\beta} + \tilde{y}_\beta \tilde{p}_{x_\beta}}{2}\rangle^2} = \frac{1}{2}\langle I_{xy}\rangle, && (20)\\
L &\equiv \langle \tilde{x}_\beta \tilde{p}_{y_\beta}\rangle - \langle \tilde{y}_\beta \tilde{p}_{x_\beta}\rangle = \langle L_z\rangle. && (21)
\end{aligned}
$$

Using the transformations in Eqs. (6,7), these emittances can be expressed in the lab-frame as well.

3. 6D ionization cooling in a bent-solenoid channel

In order to increase the phase-space density and decrease the phase-space volume occupied by a beam, non-Hamiltonian forces have to be applied. In ionization cooling channels, muons' interactions with materials result in dissipative and diffusive forces on the muons. The equations of motion written in the Larmor frame are

$$
\begin{aligned}
\frac{d\tilde{x}}{ds} &= \tilde{P}_x, && (22)\\
\frac{d\tilde{P}_x}{ds} &= -K(s)\tilde{x} + \frac{\delta\cos[\theta(s)]}{\rho(s)} - \eta(s)\left[\tilde{P}_x - \hat{b}_s(s)\tilde{y}\right] + \sqrt{\chi(\tilde{x},s)}\,\xi_x^{\mathrm{MS}}(s), && (23)\\
\frac{d\tilde{y}}{ds} &= \tilde{P}_y, && (24)\\
\frac{d\tilde{P}_y}{ds} &= -K(s)\tilde{y} + \frac{\delta\sin[\theta(s)]}{\rho(s)} - \eta(s)\left[\tilde{P}_y + \hat{b}_s(s)\tilde{x}\right] + \sqrt{\chi(\tilde{x},s)}\,\xi_y^{\mathrm{MS}}(s), && (25)\\
\frac{dz}{ds} &= \delta - \frac{\tilde{x}\cos[\theta(s)] + \tilde{y}\sin[\theta(s)]}{\rho(s)}, && (26)\\
\frac{d\delta}{ds} &= -V(s)z - (\partial_{\tilde{x}}\eta)\tilde{x} + \sqrt{\chi_\delta(\tilde{x},s)}\,\xi_z^{\mathrm{ES}}(s). && (27)
\end{aligned}
$$

Here the terms without η and χ are the direct results of Hamiltonian equations. The $\partial_{\tilde{x}}\eta$ term gives position-dependent momentum loss due to wedged absorbers in the $\tilde{x}$ direction. (We assume this wedge orientation for simplicity.) The $\hat{b}_s$ terms are because energy loss is proportional to the kinematic momentum instead of the canonical momentum. $\xi_x^{\mathrm{MS}}(s)$, $\xi_y^{\mathrm{MS}}(s)$, and $\xi_z^{\mathrm{ES}}(s)$ are uncorrelated unit stochastic quantities describing the fluctuating forces due to multiple scattering and energy straggling, respectively.

From the linear equations of motion, second-order beam-moment equations can be derived. There are 21 first-order differential equations for the change rates of the 21 second-order beam moments in 6D phase space. To understand cooling dynamics it is important to solve the moment equations analytically. However, it is a formidable task to solve such a large set of coupled differential equations. The key to success is that we can treat the muons' interactions with materials as small perturbations. Without materials, the three degrees of freedom can be decoupled as described earlier. With materials, we can still treat them as decoupled approximately, and the beam envelope functions can be approximated by their zeroth-order solution. The main effects of the material are on the emittances. Due to the dissipative and diffusive forces, the emittances will no longer be conserved. Using the above approximations, emittance evolution equations can be derived from the beam moment equations. We will not present the detailed calculations, but give the results here.

$$\begin{aligned}
\varepsilon_x' &= -[\eta - (\partial_x\eta)D_x]\varepsilon_x + \frac{1}{2}\eta\hat{b}_s\beta_T L + \frac{1}{2}\beta_T\chi + \frac{1}{2}\mathcal{H}_x\,\chi_\delta, && (28)\\
\varepsilon_y' &= -\eta\varepsilon_y + (\partial_x\eta)D_y\varepsilon_{xy} + \frac{1}{2}\left[\eta\hat{b}_s\beta_T + (\partial_x\eta)(\beta_T D_y' + \alpha_T D_y)\right]L + \frac{1}{2}\beta_T\chi + \frac{1}{2}\mathcal{H}_y\,\chi_\delta, && (29)\\
\varepsilon_z' &= -(\partial_x\eta)D_x\varepsilon_z + \frac{1}{2}\beta_L\chi_\delta + \frac{1}{2}\gamma_L(D_x^2 + D_y^2)\chi, && (30)\\
\varepsilon_{xy}' &= -[\eta - \frac{1}{2}(\partial_x\eta)D_x]\varepsilon_{xy} + \frac{1}{2}(\partial_x\eta)D_y\varepsilon_x + \frac{1}{4}\partial_x\eta(\beta_T D_x' + \alpha_T D_x)L + \frac{1}{2}\mathcal{H}_{xy}\,\chi_\delta, && (31)\\
L' &= -[\eta - \frac{1}{2}(\partial_x\eta)D_x]L + \eta\hat{b}_s\beta_T(\varepsilon_x + \varepsilon_y) + \partial_x\eta(\beta_T D_y' + \alpha_T D_y)\varepsilon_x \\
&\quad -\partial_x\eta(\beta_T D_x' + \alpha_T D_x)\varepsilon_{xy} + \mathcal{H}_L\chi_\delta, && (32)
\end{aligned}$$

where the $\mathcal{H}$ terms are the corresponding phase-space invariants formed with dispersion functions. For example, the familiar $\mathcal{H}_x$ is given by $I_x(D_x, D_x') = \gamma_T D_x^2 + 2\alpha_T D_x D_x' + \beta_T D_x'^2$. We remark that, because it is difficult to decouple the full Hamiltonian that contains the Hamiltonian part of the dissipative force, we did not separate the dissipative force into Hamiltonian part and non-Hamiltonian part as described in Ref. [11]. Therefore the coefficient matrix of the above equations is not symmetric. If we ignore the Hamiltonian part of the dissipative force, the emittance evolution equations can be obtained simply by symmetrizing the coefficient matrix of the above equations.

These equations clearly demonstrate the cooling mechanisms for the transverse and longitudinal degrees of freedom. Without the wedged absorbers, the transverse degrees of freedom all have exponential damping with a rate η, while the longitudinal degree of freedom has no damping at all. By introducing both dispersion and wedged absorbers, an exponential damping term of rate $(\partial_x\eta)D_x$ is created for the longitudinal emittance at the expense of reducing the transverse damping rate by the same amount, which is the essence of emittance exchange. In addition to these apparent damping terms, there are couplings among these modes due to the absorbers. More importantly, there are heating terms due to multiple scattering χ and energy straggling χ_δ. It is instructive to illuminate the mechanism of these heating terms. There are three types of heating terms: $\beta\chi$, $\mathcal{H}\,\chi_\delta$, and $\gamma_L D^2\chi$; all are due to noise contribution to the beam invariants. Let us first look at the three $\beta\chi$ terms. Multiple scattering and energy straggling cause the transverse and longitudinal momenta to fluctuate with $\langle p_x^2\rangle = \langle p_y^2\rangle = \chi$ and $\langle\delta^2\rangle = \chi_\delta$, but there are no correlated position fluctuations. Thus they contribute to the invariants only through the $\beta\langle p^2\rangle$ term and yield the $\beta_L\chi$ term in the two transverse emittances and the $\beta_L\chi_\delta$ term in the longitudinal emittance. In addition, the transverse momentum fluctuations cause fluctuation in the longitudinal position (but not energy) through D_x and D_y in Eq. (7), thus they contribute the term $\gamma_L(D_x^2 + D_y^2)\chi$ to the longitudinal emittance. There are four $\mathcal{H}\,\chi_\delta$ terms in the transverse emittances. Energy straggling results in energy fluctuation with $\langle\delta^2\rangle = \chi_\delta$, which leads to correlated fluctuations in both position and momentum of the transverse phase space through the dispersion D_x and D_x'. Thus it contributes to all transverse invariants and results in the four $\mathcal{H}\,\chi_\delta$ terms. The expressions of $\mathcal{H}$'s are just a reflection of the invariant structures.

To maximize the exchange rate, wedged absorbers should be placed at dispersion maximum, which means $(\partial_x\eta)$ is nonzero only around $D_{x,y}' = 0$. Furthermore, the maximum dispersion should be at minimum beta, i.e., $\alpha_T = 0$ at the wedges. Therefore, the term $\partial_x\eta(\beta_T D_y' + \alpha_T D_y)$ is approximately zero for efficient emittance exchange. Dropping this term from the above equations we obtain

$$\begin{aligned}
\varepsilon_x' &= -[\eta - (\partial_x\eta)D_x]\varepsilon_x + \frac{1}{2}\eta\hat{b}_s\beta_T L + \frac{1}{2}\beta_T\chi + \frac{1}{2}\mathcal{H}_x\,\chi_\delta, && (33)\\
\varepsilon_y' &= -\eta\varepsilon_y + (\partial_x\eta)D_y\varepsilon_{xy} + \frac{1}{2}\eta\hat{b}_s\beta_T L + \frac{1}{2}\beta_T\chi + \frac{1}{2}\mathcal{H}_y\,\chi_\delta, && (34)\\
\varepsilon_z' &= -(\partial_x\eta)D_x\varepsilon_z + \frac{1}{2}\beta_L\chi_\delta + \frac{1}{2}\gamma_L(D_x^2 + D_y^2)\chi, && (35)\\
\varepsilon_{xy}' &= -[\eta - \frac{1}{2}(\partial_x\eta)D_x]\varepsilon_{xy} + \frac{1}{2}(\partial_x\eta)D_y\varepsilon_x + \frac{1}{2}\mathcal{H}_{xy}\,\chi_\delta, && (36)\\
L' &= -[\eta - \frac{1}{2}(\partial_x\eta)D_x]L + \eta\hat{b}_s\beta_T(\varepsilon_x + \varepsilon_y) + \mathcal{H}_L\chi_\delta. && (37)
\end{aligned}$$

Note that the longitudinal equation can be easily integrated and yields an explicit formula for the longitudinal emittance. The rest of the equations are difficult to solve analytically. One way around this difficulty is to average the right-hand sides of the equations over one period of a cooling channel, and then solve them by straightforward diagonalization. This should be a good approximation because the emittances do not change much in any given period. Here we will not get into the details of solving these equations. Even without further integration, these equations tell us a great deal about how the ionization cooling works and what is necessary to limit the heating effects.

Now let us consider a few important special cases. In the case of transverse cooling in straight solenoid channels, there are no dispersions and wedged absorbers. Then the above equations reduce to [12]

$$\begin{aligned}
(\varepsilon_x + \varepsilon_y + L)' &= -\eta(\varepsilon_x + \varepsilon_y + L) + \eta \hat{b}_s \beta_T (\varepsilon_x + \varepsilon_y + L) + \beta_T \chi, && (38)\\
(\varepsilon_x + \varepsilon_y - L)' &= -\eta(\varepsilon_x + \varepsilon_y - L) - \eta \hat{b}_s \beta_T (\varepsilon_x + \varepsilon_y - L) + \beta_T \chi, && (39)\\
(\varepsilon_x - \varepsilon_y)' &= -\eta(\varepsilon_x - \varepsilon_y), && (40)\\
\varepsilon'_{xy} &= -\eta \varepsilon_{xy}, && (41)\\
\varepsilon'_z &= \frac{1}{2}\beta_L \chi_\delta. && (42)
\end{aligned}$$

These much simplified equations are fully decoupled and thus can be integrated easily. The longitudinal emittance grows linearly due to the heating from energy straggling, the asymmetric emittances are exponentially damped out, and the symmetric emittance and the angular momentum are damped to certain equilibriums determined by the balance between heating and cooling. If the beam is cylindrically symmetric, $\varepsilon_x = \varepsilon_y$ and $\varepsilon_{xy} = 0$. Then only the first two transverse

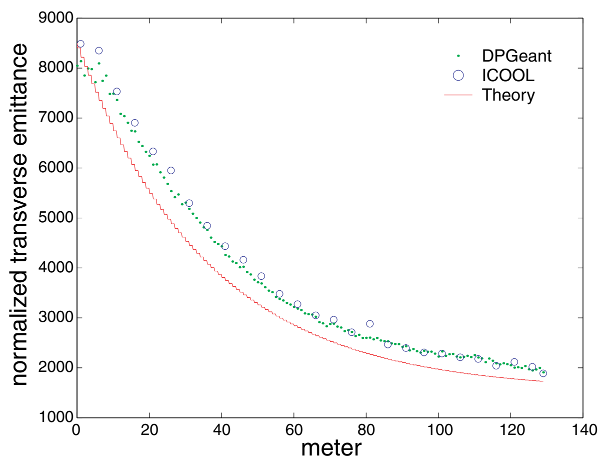

Figure 2: Transverse cooling calculated analytically, with ICOOL, and with DPGeant. (Simulation data courtesy of P. Lebrun. ©APS, PRL [13].)

equations are necessary to describe the transverse cooling. This special case has been explored analytically in detail and the result agree well with simulations [13]. In Fig. 2, we reproduced the figure showing the transverse cooling performance calculated with two simulation codes and our analytical solution for a FOFO cooling channel.

Furthermore, if we drop the angular momentum and the ε_{xy} terms, Eqs. (28,29,30) describe the ionization cooling in a quadrupole channel as well. Detailed solution of 6D cooling in quadrupole channels can be found in Ref. [14]. Without dispersions, the transverse cooling equation reduces to the simple form [15]

$$\varepsilon'_x = -\eta \varepsilon_x + \frac{1}{2}\beta_x \chi. \tag{43}$$

This well-known equation has been a major guide for cooling channel designs.

4. Status of cooling channel developments

Ionization cooling channels have been under intense investigation over the last several years. Because of the difficulty in longitudinal cooling and the exciting applications of neutrino factories, which do not require longitudinal cooling, focus has been temporarily shifted to transverse cooling channel development in the last couple of years. Two intensive six-month studies on the feasibility of neutrino factories were carried out at Fermi National Accelerator Laboratory and at Brookhaven National Laboratory [6, 7]. Now a few designs have been worked out for transverse ionization cooling in straight solenoidal channels. Although the cooling performance is not impressive yet, moderate transverse cooling has been demonstrated with two independent simulation codes. The simulation results agree with each other and with the linear theory discussed above [13]. Thus it is reasonably safe to say that these designs are physically sound. These channels are also designed to be feasible from an engineering standpoint. Now the transverse cooling channel development reaches the stage of engineering testing of hardware components and, hopefully with adequate funding, experimental demonstration of transverse ionization cooling.

The longitudinal ionization cooling problem is much harder. The linear 6D ionization cooling theory has just been worked out. Although there are a few promising candidates, no design has been demonstrated physically sound in reliable simulations. Much R&D effort will be required to tackle this roadblock for muon colliders.

As a final remark, we believe that the physical design of ionization cooling channels is still far from mature, and there is potential for major advancements, for which much research is needed.

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References

[1] M. Tigner, Physics Today, **54**(1), 36 (2001).
[2] C. Ankenbradt et al., Phys. Rev. ST Accel. Beams **2**, 081001 (1999).
[3] A. Chao and M. Tigner (eds.), *Handbook of Accelerator Physics and Engineering*, World Scientific Publishing Co., p. 29 (1999).
[4] A.M. Sessler, Physics Today, **51**(3), 48 (1998).
[5] S. Geer, Phys. Rev. D **57**, 6989-97 (1998).
[6] "A Feasibility Study of a Neutrino Source Based on a Muon Storage Ring," edited by N. Holtkamp and D. Finley, 2000.
[7] "Feasibility Study-II of a Muon-based Neutrino Source," edited by S. Ozaki, R.B. Palmer, M.S. Zisman, and J.C. Gallardo, 2001 (http://www.cap.bnl.gov/mumu/studyii/final_draft/The-Report.pdf).
[8] C. Caso et al., The European Physical Journal C**3**(1), 146 (1998).
[9] D. Neuffer, "μ^+-μ^- COLLIDERS," CERN 99-12 (1999).
[10] Chun-xi Wang, Proceedings of NuFact'01 (2001).
[11] F. Ruggiero, E. Picasso, and L.A. Radicati, Ann. of Phys. **197**, 396-438 (1990).
[12] Chun-xi Wang and Kwang-Je Kim, Proceedings of the 18th International Conference on High Energy Accelerators, Japan, 2001.
[13] Kwang-Je Kim and Chun-xi Wang, PRL **85**(4) 760 (2000).
[14] Chun-xi Wang, Proceedings of NuFact'01 (2001).
[15] D. Neuffer, Part. Acc. **14**, 75 (1983).