A Vlasov-Fokker-Planck Modeling of Intrabeam Scattering and Wake Forces for Electron Beams

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We present a model to study in a self-consistent way the interplay between intrabeam scattering and wake-field forces in low-emittance high-intensity electron storage rings. The regime of interest is that of the damping rings for the next generation of linear colliders.

1. Introduction

In low emittance electron rings the longitudinal dynamics may be significantly affected by both intrabeam scattering (IBS) and wake-field effects. In particular, this will be the case in the damping rings for the next generation of linear colliders [1]. Because both effects depend on and in turn contribute to determining the beam distribution in phase space a self-consistent treatment is called for. A suitable framework is given by the Vlasov-Fokker-Planck (VFP) equation. While the VFP equation already represents a standard tool for investigating beam dynamics, inclusion of intrabeam scattering in combination with radiation and wake-field effects does not appear to have been considered before. Our goal in this paper is to provide this extension. In plasma physics a Fokker-Planck modelling of interparticle collisions has long been established. Here we recall how to adapt this description to particle beams in the full 6D phase space and we then derive a reduced 1D VFP equation for the sole longitudinal motion. In this form the problem can be studied by numerically solving the reduced PDE for the longitudinal beam distribution and two ordinary differential equations for the evolution of the transverse emittances. A code for finding the solutions of the reduced problem is currently under development and will be used to study equilibrium distributions at high current, the effects of IBS on the onset of microwave instability and beam dynamics above the instability threshold.

2. The VFP Equation

Our model of beam dynamics is the equation

$$\frac{\partial f}{\partial s} + \{f, H\} = \left(\frac{\partial f}{\partial s}\right)_c + \mathrm{FP}_{\mathrm{rad}}(f) \tag{1}$$

obeyed by the beam distribution function in the 6D phase space $f = f(\boldsymbol{X}; s)$, with $\boldsymbol{X} = (x, p_x, y, p_y, z, p_z)$. The first two pairs of canonical coordinates, are relative to the motion in the horizontal and vertical planes, while z describes the longitudinal displacement with respect to the synchronous particle and $p_z = \Delta p/p$ is the relative deviation of the total momentum from the design value; s is the independent 'time-like' variable giving the location of a particle along the lattice. The Hamiltonian H may include wake-field and possibly space charge forces in addition to the external forces provided by the magnetic lattice and RF cavities; $\{\cdot, \cdot\}$ are the Poisson brackets. The first term on the RHS represents the effect of collisions and the second that of synchrotron radiation. For the collision term we use a Fokker-Planck approximation (which can be obtained from the Boltzmann collision integral by doing a small angle expansion and retaining

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the lowest order terms [2]), $(\partial f/\partial \tilde{t})_c \simeq \tilde{\mathrm{FP}}_c(\boldsymbol{X})$ with $(i,j = x,y,z)$

$$\tilde{\mathrm{FP}}_c(\boldsymbol{X}) = -\sum_i \frac{\partial}{\partial \tilde{p}_i}(\tilde{f}\tilde{D}_i) + \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial \tilde{p}_i \partial \tilde{p}_j}(\tilde{f}\tilde{D}_{ij}). \tag{2}$$

The tilde ~denotes quantities in the beam rest frame; in this frame $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}}) \equiv \tilde{\boldsymbol{X}}$ are the actual position and mechanical momentum. Drift $\tilde{D}_i$ and diffusion $\tilde{D}_{ij}$ coefficients can be written as

$$\begin{aligned} \tilde{D}_i(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}}) &= \Gamma \frac{\partial}{\partial \tilde{p}_i}\int \frac{\tilde{f}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}}')}{|\tilde{\boldsymbol{p}} - \tilde{\boldsymbol{p}}'|} d^3\tilde{\boldsymbol{p}}', & (3)\\ \tilde{D}_{ij}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}}) &= \frac{\Gamma}{2}\frac{\partial^2}{\partial \tilde{p}_i \partial \tilde{p}_j}\int \tilde{f}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}}')|\tilde{\boldsymbol{p}} - \tilde{\boldsymbol{p}}'| d^3\tilde{\boldsymbol{p}}', & (4) \end{aligned}$$

where $\Gamma = 8\pi m^3 c^4 r_c^2 \log\Lambda_c$, with r_c being the classical radius of the particle and $\log\Lambda_c \simeq \log(2/\theta_m)$ is the so called Coulomb logarithm. For emittance dominated beams the minimum scattering angle θ_m is determined by bunch sizes. The integrals in (3) and (4) are known in plasma physics as 'Rosenbluth potentials' although it appears they were first derived by Landau.

To obtain the Fokker-Planck equation in the lab frame it is just a matter of applying the proper transformation to Equation (2). In the paraxial approximation the transformation from the lab frame coordinates $\boldsymbol{X}$ to the beam frame coordinates $\tilde{\boldsymbol{X}}$ is just a scaling, which can be represented by a diagonal matrix M, $\tilde{\boldsymbol{X}} = \mathrm{M}\boldsymbol{X}$. Invariance of the number of scattered particles as recorded in the two frames $\mathrm{FP}_c(\boldsymbol{X})d^6\boldsymbol{X}ds = \tilde{\mathrm{FP}}_c(\tilde{\boldsymbol{X}})d^6\tilde{\boldsymbol{X}}d\tilde{t}$ permits writing the Fokker-Planck term in the lab frame in terms of (2): $\mathrm{FP}_c(\boldsymbol{X}) = \tilde{\mathrm{FP}}_c(\mathrm{M}\boldsymbol{X})/|\det \mathrm{M}|\gamma_0 v_0$, having used $ds = v_0 dt$, where v_0 is the design beam velocity in the lab frame, and the relativistic time dilation $dt = \gamma d\tilde{t}$. If we denote as N the part of transformation M relative to the momenta only we have $\mathrm{N}_{11} = \mathrm{N}_{22} = p_0$ and $\mathrm{N}_{33} = p_0/\gamma_0$ (off-diagonal terms vanish) and the VFP equation in the Lab frame can be written as

$$\frac{\partial f}{\partial s} + \{f, H\} = -\sum_i \frac{\partial}{\partial p_i}(fD_i) + \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial p_i \partial p_j}(fD_{ij}), \tag{5}$$

with

$$\begin{aligned} D_i &= \frac{\Gamma}{\gamma_0^2 v_0 \mathrm{N}_{ii}^2}\frac{\partial}{\partial p_i}\int \frac{f(\boldsymbol{x}, \boldsymbol{p}')}{|\mathrm{N}(\boldsymbol{p} - \boldsymbol{p}')|} d^3\boldsymbol{p}', \\ D_{ij} &= \frac{\Gamma}{2\gamma_0^2 v_0 \mathrm{N}_{ii}^2 \mathrm{N}_{jj}^2}\frac{\partial^2}{\partial p_i \partial p_j}\int f(\boldsymbol{x}, \boldsymbol{p}')|\mathrm{N}(\boldsymbol{p} - \boldsymbol{p}')| d^3\boldsymbol{p}', \end{aligned}$$

having also made use of $f(\boldsymbol{X}) = |\det \mathrm{M}|\tilde{f}(\mathrm{M}\boldsymbol{X}) = p_0^3\tilde{f}(\mathrm{M}\boldsymbol{X})$. For a discussion on the range of applicability of the VFP equation we refer the reader to *e.g.* [2]. The familiar IBS growth rates for the emittances [3] can obtained after multiplying both sides of Eq. (5) by second powers of momenta and integrating over the phase space variables under the assumption that the distribution function f is gaussian.

3. The Reduced 1-D VFP Equation

For the beam we assume a distribution of the form

$$f(\boldsymbol{X}) = \frac{Ng(z, p_z)}{(2\pi)^2\varepsilon_x\varepsilon_y}\exp\left[-S^x(\boldsymbol{X}) - S^y(\boldsymbol{X})\right], \tag{6}$$

with the horizontal and linear invariants given by

$$\begin{aligned} S^x = {} & [\beta_x(p_x - p_z\eta_x')^2 + 2\alpha_x(x - p_z\eta_x)(p_x - p_z\eta_x') \\ & + \gamma_x(x - p_z\eta_x)^2]/2\varepsilon_x. \end{aligned} \tag{7}$$

(Similar expression for S_y). Here α_x, β_x, γ_x are the Courant functions and η_x is the dispersion function. Let the Hamiltonian H_z for the longitudinal motion be that of an ultra-relativistic electron bunch experiencing linear RF forces and single-turn wake fields. Such a Hamiltonian reads [4]

$$
\begin{aligned}
H_z &= \frac{1}{2}p_z^2\alpha_c + \frac{1}{2\alpha_c}\left(\frac{\nu_s}{R}\right)^2 z^2 + \\
& \quad I\int_z^\infty dz''\int_{-\infty}^\infty dz' W(z''-z')\rho_z(z),
\end{aligned}
\tag{8}
$$

where $W(z''-z')/2\pi R$ has the meaning of averaged (over one turn) longitudinal electric field per unit charge acting on a test particle in z'' due to a point source at z'; R is the machine radius; $\rho_z = \int dp_z g(z,p_z)$ is the longitudinal beam density; ν_s the synchrotron oscillation tune; α_c the momentum compaction and finally $I = e^2N/2\pi Rcp_0$. The reduced VFP equation obeyed by g is obtained from Equation (5) by integrating with respect to the transverse coordinates. This leads to

$$
\frac{\partial g}{\partial s} + \{g, H_z\} = -\frac{\partial}{\partial p_z}(g\hat{D}_z) + \frac{1}{2}\frac{\partial^2}{\partial p_z^2}(g\hat{D}_{zz}) + \mathrm{FP}_{\mathrm{rad}}(g),
\tag{9}
$$

where the drift and diffusion coefficients due to IBS are

$$
\hat{D}_z = \int d\boldsymbol{x}_\perp d\boldsymbol{p}_\perp f(\boldsymbol{X}) D_z(\boldsymbol{x},\boldsymbol{p}),
\tag{10}
$$

$$
\hat{D}_{zz} = \int d\boldsymbol{x}_\perp d\boldsymbol{p}_\perp f(\boldsymbol{X}) D_{zz}(\boldsymbol{x},\boldsymbol{p}),
\tag{11}
$$

with D_z and D_{zz} given by the expressions at the end of the previous Section. The last term on the RHS of (9) represents the Fokker-Planck term associated with the effects of radiation on the longitudinal motion. It can be written as

$$
\mathrm{FP}_{\mathrm{rad}}(g) = \frac{2}{c\tau_p^{\mathrm{rad}}}\frac{\partial}{\partial p_z}\left(p_z g + \sigma_{p0}^2\frac{\partial}{\partial p_z}g\right),
\tag{12}
$$

where τ_p^{rad} is the longitudinal damping time and σ_{p0} the natural (relative) momentum spread of a bunch at equilibrium due to sole radiation effects. The results of integration in (10) and (11) are best written in terms of the auxiliary function

$$
\mathcal{F}_\alpha(u) = \int_0^\infty d\lambda \frac{\lambda^\alpha e^{-[A_{33}+\lambda-A_{13}/(A_{11}+\lambda)]u^2/4}}{(\lambda+\beta_x/\varepsilon_x)^{\frac{1}{2}}(\lambda+\beta_y/\varepsilon_y)^{\frac{1}{2}}}.
\tag{13}
$$

After defining $A = Nr_c^2\log(\Lambda_c)/2\sqrt{\pi}\beta_0^4\gamma_0^5\varepsilon_x\varepsilon_y$ and $u_z = (p_z - p_z')/\gamma_0$ we have

$$
\begin{aligned}
\hat{D}_z &= -A\int_{-\infty}^\infty dp_z' g(z,p_z') u_z \mathcal{F}_{\frac{1}{2}}(u_z), \\
\hat{D}_{zz} &= \frac{A}{2}\int_{-\infty}^\infty dp_z' g(z,p_z')[2\mathcal{F}_{-\frac{1}{2}}(u_z) - u_z^2\mathcal{F}_{\frac{1}{2}}(u_z)].
\end{aligned}
$$

with $A_{11} = \beta_x\varepsilon_x$, $A_{122} = \beta_y\varepsilon_y$, $A_{33} = \mathcal{H}_x/\varepsilon_x$ and $A_{13} = -(\beta_x\eta_x' + \alpha_x\eta_x)\varepsilon_x$. The RHS of the above expressions are understood to be averaged over the lattice.

Equation (9) is not self-contained because the IBS drift and diffusion coefficients depend on the transverse emittances. The evolution of the transverse emittances can be determined by solving a pair of ODE's using the familiar expressions for the IBS growth [3], radiation damping and excitation rates. At each time step these equations require specification of longitudinal rms bunch size and momentum spread, which are determined from the solution g of (9). Notice that this scheme is not completely self-consistent because the familiar IBS growth rates are derived on the assumption that the beam distribution is gaussian while g, in general, is not.

To solve the VFP equation (9) one can follow the method proposed in [4]. An implementation of this method for this problem is currently under development. The distribution function g is represented on a cartesian grid and operator splitting is used to evaluate the propagation of g under the action of the Vlasov and FP part of the equation separately. The Hamiltonian part of the dynamics in particular is treated by the method of the Perron-Frobenius operator. For some preliminary results see [5].

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