An Averaged Vlasov Equation for the Strong–Strong Beam–Beam

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Abstract

The Strong–Strong Beam–Beam is studied in the framework of maps with a “Kick–Rotate” model. The classical method of averaging is applied to derive an approximate map which is equivalent to a flow within the averaging approximation. This flow leads to an averaged Vlasov equation (AVE); the basic model of this paper. The power of this approach is evidenced by the fact that the AVE has exact equilibria and the associated linearized equations have uncoupled azimuthal Fourier modes. In the usual way, the Fourier mode equations lead to a Fredholm integral equation of the third kind. We have solved this numerically in a special case, found the $\sigma$ and $\pi$ mode frequencies and they are in excellent agreement with simulation.

1 INTRODUCTION

In this paper we introduce a new approximate model for the strong–strong beam–beam interaction. This both generalizes and simplifies the work of [1, 2, 3] on the strong–strong beam–beam interaction in high energy colliders. Our model is based on the classical method of averaging generalized to maps and collective forces and leads to an averaged Vlasov equation (AVE). The previous pioneering works make several assumptions and approximations which are difficult to assess mathematically. Here we make only one assumption, we assume that the classical method of averaging is valid for this problem. We do not introduce a delta function into the Vlasov equations of motion, which is ambiguous because the density is discontinuous at the point of the delta function. Thus we have no need to smooth the delta function, an approximation, the nature of which is hard to assess. The technique we introduce should be of general interest to studies of the Vlasov equation with perturbative collective force.

To motivate the method of averaging for maps with a collective force we review our approach to rigorous averaging. For example, they have equilibrium solutions, and this is discussed in Section 5. This suggests the exact model has quasi–equilibria. We have compared one of the equilibria with the simulation for that initial density and we obtain excellent agreement over 130,000 turns. Because we have equilibria, we can linearize about these equilibria and study the linearized equations. This is done in Section 6, where we introduce the equations for the $\sigma$ and $\pi$ modes. Unlike previous approaches, these equations have uncoupled Fourier modes in the azimuthal variable. Using the usual ansatz, we derive an integral equation of the third kind for the $\sigma$ and $\pi$ mode frequencies. We have solved the equations for the dipole Fourier mode, and obtain excellent agreement with simulation for both the $\sigma$ and $\pi$ mode frequencies. Section 7 discusses our plans for the future. The appendix contains three topics which are germane to our discussions. The Gronwall inequality which is a basic inequality of the elementary theory of ODEs, the Besjes Lemma which is the basic tool in our approach to obtaining rigorous error bounds in our averaging procedures and finally a statement of the Birkhoff Ergodic Theorem which we need for the existence of our averaged problem.

2 AVERAGING PROCEDURE AND ERROR BOUNDS

Here we show how the method of averaging can be applied to a perturbed autonomous linear system defined by a matrix $A$. We use $t$ as the independent variable, but of course it could refer to distance $s$ or azimuthal variable $\theta$ for example.

Consider the IVP

$$\frac{d}{dt} \tilde{x} = A \tilde{x} + \epsilon \tilde{f}(\tilde{x}, t) + O(\epsilon^2) ; \quad \tilde{x}(0) = \tilde{x}_0 ,$$

(1)

where $f(x, \cdot)$ is quasiperiodic. The first step is to transform it to slowly varying coordinates. It is natural to use the “variation of parameters” transformation given by

$$\tilde{x} = e^{At} \tilde{z} .$$

(2)

This leads to

$$\frac{d}{dt} \tilde{z} = \epsilon e^{-At} \tilde{f}(e^{At} \tilde{z}, t) + O(\epsilon^2) := \epsilon \tilde{g}(\tilde{z}, t) + O(\epsilon^2) ,$$

(3)

which is in a standard form for averaging. The averaged IVP is given by

$$\frac{d}{dt} \tilde{v} = \epsilon \tilde{g}(\tilde{v}) ; \quad \tilde{v}(0) = \tilde{x}_0 ,$$

(4)
where

\[ \bar{g}(\bar{v}) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \bar{g}(\bar{v}, t) \, dt . \]  

The averaging formalism is now complete and we turn our attention to the error analysis. We subtract (4) from (3) and integrate to obtain

\[ \bar{z}(t) - \bar{v}(t) = e \int_0^t \left[ \bar{g}(\bar{z}(t'), t') - \bar{g}(\bar{v}(t'), t') \right] \, dt' + \frac{e^2}{2} \int_0^t \left[ \bar{g}(\bar{v}(t'), t') - \bar{g}(\bar{v}(t')) \right] \, dt' + \frac{1}{2} \bar{g}(\bar{v}) + O(e^2) . \]  

(6)

The second integral on the r.h.s. has zero mean and by the Besjes Lemma (see appendix) it can be bounded by a constant, say \( C_1 \) on \( O(1/e) \) \( t \)-intervals, in the case where, for example, \( g \) is quasiperiodic. Thus we obtain

\[ |\bar{z}(t) - \bar{v}(t)| \leq e L \int_0^t |\bar{z}(t') - \bar{v}(t')| \, dt' + e(C_1 + C_2) , \]

for \( 0 \leq t \leq T/e \), where \( L \) is the Lipschitz constant for \( g \).

The Gronwall inequality (see appendix) then gives

\[ |\bar{z}(t) - \bar{v}(t)| \leq e C e^{LT} ; \quad 0 \leq t \leq T/e , \]

where \( C = C_1 + C_2 \) and \( C_2 \) came from the \( O(e^2) \) term in (1). Thus we say that the solution of (4) gives an \( O(e) \) approximation to the solution of (3) on \( O(1/e) \) \( t \)-intervals. We call this a formal error analysis because solution domains and associated constants must be carefully defined and resonance considerations addressed. Reference [4] discusses this in detail and an improved approach for maps is given in [5].

3 THE KICK-ROTATE MODEL IN 2-D PHASE SPACE

Here we consider head on collisions of two counter rotating bunches with one IP and with the betatron motion modeled by a rotation with tune \( Q_0 = \pi/2 \). Our viewpoint is directly before the IP and \( \psi_n(\bar{x}) \) and \( \psi_n^*(\bar{x}) \) denote the phase space densities of the two bunches just before the \( n \)-th passage bunch crossing at the IP. All densities are normalized to 1. In general, any symbol without a star will refer to one (the "unstarred") beam and any symbol with a star will refer to the other (the "starred") beam.

The beam–beam kick is written

\[ \bar{x} \mapsto \bar{x}_n := \left( \begin{array}{c} q_n \\ p_n - \zeta(g \ast \psi_n^*)(\bar{x}) \end{array} \right) = \bar{x}_n , \]

where

\[ (g \ast \psi)(\bar{x}) := \int_{\mathbb{R}^2} g(q - q') \psi(q) \, dq' \, dx . \]

and \( g(q) \) is the derivative of the Green's function of the appropriate Poison problem. For example,

\[ \text{CR: } g(q) \propto \text{sgn}(q) , \quad \text{YKZ: } g(q) \propto p.V. \frac{1}{q} . \]

(11)

where CR refers to the case discussed in [1] and YKZ refers to the case discussed in [2]. Equation (9) is the kick on the unstarrred beam due to the starred beam. The perturbation parameter \( \zeta \) is proportional to the the linear beam–beam tune shift parameter as will be seen in Eq. (43). For the propagation through a linear lattice we assume linear normal form coordinates, thus

\[ \bar{x}_n' \mapsto \bar{R}\bar{x}'_n := \bar{x}_{n+1} , \quad \bar{R} = \left( \begin{array}{cc} \cos \mu & \sin \mu \\ -\sin \mu & \cos \mu \end{array} \right) = e^{i\zeta} \]

(12)

where \( \mu = 2\pi Q \) is the phase advance per turn and \( \bar{R} \) is the symplectic unit matrix.

Thus the OTM becomes

\[ \bar{x}_{n+1} = \bar{R} \left[ \bar{x}_n - \zeta \left( \begin{array}{c} 0 \\ 1 \end{array} \right) (g \ast \psi_n^*)(\bar{x}^*_n) \right] , \]

(13)

and the turn–by–turn evolution of the densities is given by

\[ \psi_{n+1}(\bar{x}_{n+1}) = \psi_n(\bar{x}_n) \ast \psi_{n+1}^*(\bar{x}^*_n) . \]

(14)

We assume in the following that \( Q_0 = Q_{0'} \) and \( \zeta = \zeta^* \).

![Figure 1: The kick rotate model](image-url)
(g * ψ_n^{*})(x(Θ_n, J_n)) + O(ζ^2) \quad (16) \]

\begin{align*}
J_{n+1} &= J_n - \zeta \sqrt{2J_n} \cos(nμ + Θ_n) \times \\
& \quad (g * ψ_n^{*})(x(Θ_n, J_n)) + O(ζ^2). \quad (17)
\end{align*}

We note that for small ζ the angle Θ and action J are slowly varying. Introducing the slowly varying density by

\[ Ψ_n(Θ, J) := ψ_n(\sqrt{2J} \sin(nμ + Θ), \sqrt{2J} \cos(nμ + Θ)), \quad (18) \]

we can rewrite the convolution \( g * ψ_n \) as an integral over the slow variables as

\[ (g * ψ_n^{*})(x(Θ_n, J_n)) = \int_C \left( \sqrt{2J_n} \sin(nμ + Θ_n) - \sqrt{2J} \sin(nμ + Θ') \right) dP_n, \quad (19) \]

where \( C := [0, 2π] \times \mathbb{R}^+ \) is the product of the one-torus and the positive real axis and \( dP_n \) is shorthand for \( Ψ_n(Θ', J') dΘ' dJ' \). Furthermore, introducing the Green’s function \( G(q) \) via \( \frac{d}{dq} G = g \) we can rewrite the OTM defined by (16–17) compactly in terms of a map generator \( F \) as

\begin{align*}
Θ_{n+1} &= Θ_n + ζ Ω F(Θ_n, J_n, nμ; Ψ_n^{*}) + O(ζ^2), \quad (20) \\
J_{n+1} &= J_n - ζ Ω F(Θ_n, J_n, nμ; Ψ_n^{*}) + O(ζ^2), \quad (21)
\end{align*}

where

\[ F(Θ, J, κ; Ψ) := \int_C \left( \sqrt{2J} \sin(κ + Θ) - \sqrt{2J} \sin(κ + Θ') \right) dP, \quad (22) \]

and

\[ Ψ_{n+1}(Θ_{n+1}, J_{n+1}) = Ψ_n(Θ_n, J_n). \quad (23) \]

### 4 MAP–AVERAGING

The OTM (16), (17) (and (20), (21) of course) contain the slow variables \((Θ, J)\), the slow density \(Ψ\) and have an explicit time dependence through the term \(nμ\). The averaged equations could be obtained by simply averaging \( F \) in (20), (21), but in order to avoid the issue of the commutation of partial derivative and average we proceed with the explicit form of the OTM (16), (17). Thus the averaged equations are obtained by dropping the \(O(ζ^2)\) term in (16), (17) and averaging over \(n\) holding \(Θ_n, J_n\) and \(Ψ_n^{*}\) fixed, in complete analogy with Section 2.

First we note that the argument of \(g\) in (19) can be written

\[ \sqrt{2J} \sin(nμ + Θ) - \sqrt{2J} \sin(nμ + Θ') = \cos(nμ + θ(Θ, J, Θ', J')) D(J, J', Θ - Θ') \quad (24) \]

where

\[ D(J, J', θ) := \sqrt{2J + 2J' - 4\sqrt{JJ'} \cos θ}, \quad (25) \]

where \(θ\) is defined by (24) and both \(θ\) and \(D\) do not explicitly depend on \(n\). We rewrite (16), (17) as

\begin{align*}
Θ_{n+1} &= Θ_n + ζ \frac{1}{\sqrt{2J_n}} \sin(nμ + Θ_n) \times \\
& \quad \left( f(Θ_n, J_n, nμ; Ψ_n^{*}) \right), \quad (26) \\
J_{n+1} &= J_n - ζ \sqrt{2J_n} \cos(nμ + Θ_n) \times \\
& \quad \left( f(Θ_n, J_n, nμ; Ψ_n^{*}) \right), \quad (27)
\end{align*}

\[ f(Θ, J, κ; Ψ) := \int_C g(D(J, J', Θ - Θ'), \cos(κ + θ(Θ, J, Θ', J')) dP. \quad (28) \]

Clearly,

\[ e^{i(nμ + θ)} f = \int_C e^{i(θ - θ')} e^{i(nμ + θ')} g(D \cos(nμ + θ)) dP, \quad (29) \]

and for \(g\) bounded the dominated convergence theorem gives

\[ e^{i(nμ + θ)} f' = \int_C e^{i(θ - θ')} e^{i(nμ + θ')} g(D \cos(nμ + θ)) dP, \quad (30) \]

that is, we can interchange the order of taking the average and integrating. If \(μ/2π\) is irrational then Birkhoff’s ergodic theorem, [6, p. 30] (see appendix also), gives

\[ e^{i(nμ + θ)} g(D \cos(nμ + θ)) := \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{i(nμ + θ')} g(D \cos(nμ + θ)) = \]

\[ \frac{1}{2π} \int_0^{2π} g(D \cos θ) e^{iθ} dθ = \]

\[ \frac{1}{2π} \int_0^{2π} g(D \cos θ) \cos θ dθ =: \mathfrak{f}(D). \quad (31) \]

Thus we obtain the averaged OTM

\begin{align*}
Θ_{n+1} &= Θ_n + ζ \frac{1}{\sqrt{2J_n}} \times \\
& \quad \int_C \sin(Θ_n - θ) \mathfrak{f}(D) dP_n^*, \quad (32) \\
J_{n+1} &= J_n - ζ \sqrt{2J_n} \times \\
& \quad \int_C \cos(Θ_n - θ) \mathfrak{f}(D) dP_n^*. \quad (33)
\end{align*}

These can be written

\begin{align*}
Θ_{n+1} &= Θ_n + ζ Ω \mathcal{F}(Θ, J; Ψ_n^{*}), \quad (34) \\
J_{n+1} &= J_n - ζ Ω \mathcal{F}(Θ, J; Ψ_n^{*}), \quad (35)
\end{align*}

where

\[ \mathcal{F}(Θ, J; Ψ) := \int_C \mathcal{G}(D(J, J', Θ - Θ')) dP, \quad (36) \]

\[ \mathcal{G}(D) = \frac{1}{2π} \int_0^{2π} \mathcal{G}(D \cos θ) dθ. \quad (37) \]
It can be shown that \( \bar{F} \) is actually the average of \( F \), justifying the comment in the first paragraph of this section. Note the convolution structure w.r.t \( \Theta \) in (36), which will be important in what follows. In the CR case \( \bar{G}(D) = \frac{\pi}{2} D \).

The OTMs of (32), (33) and (34), (35) are only symplectic to \( O(\zeta^2) \) as is easily checked. However, we can interpret (34), (35) as an Euler–step with step length 1 of a strictly Hamiltonian system with the time independent Hamiltonian \( \zeta \bar{F}(\Theta, J, \Psi^*) \), that is,

\[
\frac{d}{dt} \Theta = \zeta \partial_J \bar{F}(\Theta, J, \Psi^*), \quad (38)
\]

\[
\frac{d}{dt} J = -\zeta \partial_\Theta \bar{F}(\Theta, J, \Psi^*). \quad (39)
\]

To see this, integrate these Hamiltonian equations from \( t = n \) to \( t = n + 1 \) and apply Taylor’s theorem to obtain (32) and (33) to \( O(\zeta^2) \). (Note that this does not work for the non-averaged OTM because the associated Hamiltonian flow is explicitly time dependent and the Euler step does not give back the OTM to \( O(\zeta^2) \)). Comparing these maps and applying a Gronwall inequality for maps much as in Section 2, shows that these two systems are \( O(\zeta) \) close on \( O(1/\zeta) \) \( n \)-intervals, which is what we expect for the relation between the exact model and the averaged model of (32) and (33). Thus we take the Hamiltonian flow and the associated Vaslov equations as our averaged model.

Scaling the independent variable by \( \tau := \zeta t \) we obtain coupled system of AVEs for \( \Psi \) and \( \Psi^* \)

\[
0 = \partial_\tau \Psi + \partial_J \bar{F}(\Theta, J; \Psi^*) \partial_\Theta \Psi - \partial_\Theta \bar{F}(\Theta, J; \Psi^*) \partial_J \Psi, \quad (40)
\]

\[
0 = \partial_\tau \Psi^* + \partial_J \bar{F}(\Theta, J; \Psi^*) \partial_\Theta \Psi^* - \partial_\Theta \bar{F}(\Theta, J; \Psi^*) \partial_J \Psi^*. \quad (41)
\]

These scaled AVEs allow an increase of the step size for numerical integration by a factor of \( O(1/\zeta) \) in comparison with the non-averaged OTM.

5 QUASI–EQUILIBRIA

Now let \( \Psi = \Psi^* = \Psi_e(J) \), then

\[
\bar{F}_e(J; \Psi_e) := \bar{F}(\Theta, J; \Psi_e)
\]

\[
= \int_0^\infty \left[ \int_0^{2\pi} \bar{G}(D(J, J', \Theta - \Theta')) d\Theta' \right] \times
\]

\[
\Psi_e(J') dJ'
\]

is independent of \( \Theta \) because of the convolution structure and because \( D(\cdot, \cdot, \cdot) \) is periodic in \( \vartheta \).

Thus \( \Psi_e(J) \) is an exact equilibrium of the averaged kick–rotate system and we may expect it to be a quasi–equilibrium of the exact system for large times.

Figure 2 shows the evolution of the action density of one of the bunches for two different initial phase space densities. The densities are plotted at a hundred out of 130,000 turns. In the case of the red crosses, the initial densities of both beams were the centered Gaussians \( \Psi^*_a(\Theta, J) = \Psi_0(\Theta, J) = \frac{\pi}{2} e^{-J^2/2} \). To the eye there is no evolution, consistent with the averaging approximation. In the case of the green \( \times \)-es, both initial densities had been given a \( \Theta \)-dependence by shifting the Gaussians by \( \pm 1 \sigma \). Here one observes an oscillating action density, again consistent with the averaging approximation.

For future reference we note that the ”weak–strong” amplitude dependent tune shift

\[
\Delta Q(J) := \frac{\zeta}{2\pi} \omega(J) := \frac{\zeta}{2\pi} \partial_J \bar{F}_e(J; \Psi_e) \quad (43)
\]

depends on the choice of the equilibrium. The linear beam–beam tune shift parameter is then \( \xi := \frac{\pi}{2\zeta} \omega(0) \).

6 THE LINEARIZED EQUATIONS

To determine the linearized equations we write

\[
\Psi(\Theta, J, \tau) = \Psi_e(J) + \Psi_1(\Theta, J, \tau) \quad (44)
\]

\[
\Psi^*(\Theta, J, \tau) = \Psi_e(J) + \Psi^*_1(\Theta, J, \tau) \quad (45)
\]

Plugging into the AVEs and dropping the nonlinear terms gives

\[
\partial_\tau \Psi_1 + \omega(J) \partial_\Theta \Psi_1 - \Psi^*_1(J) \partial_\Theta \bar{F}(\Theta, J; \Psi^*_1) = 0 \quad (46)
\]

and a corresponding equation with \( \Psi_1 \) and \( \Psi^*_1 \) interchanged. The two equations can be decoupled by introducing

\[
\Psi^\pm := \Psi_1 \pm \Psi^*_1, \quad (47)
\]

which yields

\[
\partial_\tau \Psi^\pm + \omega(J) \partial_\Theta \Psi^\pm \mp \Psi^*_1(J) \partial_\Theta \bar{F}(\Theta, J; \Psi^*_1) = 0 \quad (48)
\]

The function \( \bar{G}(D(J, J', \Theta)) \) is periodic in \( \Theta \) and can be expanded as

\[
\bar{G}(D(J, J', \Theta)) = \sum_{k \in \mathbb{Z}} G_k(J, J') e^{ik\Theta} \quad (49)
\]
Here the $G_k$ are real, symmetric in $k$ ($G_k = G_{-k}$) and symmetric under interchange of $J$ and $J'$ ($G_k(J,J') = G_k(J',J)$). Let

$$\Psi^\pm(\Theta, J, \tau) = \sum_{k \in \mathbb{Z}} \Psi^\pm_k(J, \tau) e^{i k \Theta},$$

then the convolution structure of $\tilde{F}$ w.r.t. $\Theta$ gives

$$2\pi \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^+} G_k(J, J') \Psi^\pm_k(J', \tau) dJ' e^{i k \Theta}. \quad (51)$$

Therefore the Fourier modes of $\Psi^\pm$ are automatically decoupled in first order averaging and satisfy

$$0 = \partial_\tau \Psi^\pm_k + i k \omega(J) \Psi^\pm_k$$

$$+ i k \Psi^\pm_k(J) 2\pi \int_{\mathbb{R}^+} G_k(J, J') \Psi^\pm_k(J', \tau) dJ'. \quad (52)$$

The ansatz

$$\Psi^\pm_k(J, \tau) = e^{i \Omega \tau} \sqrt{\Psi^\pm_k(J)} \Phi^\pm_k(J) \quad (53)$$

leads to the Fredholm integral equation of the third kind

$$0 = (\Omega + k \omega(J)) \Phi^\pm_k(J)$$

$$+ 2\pi k \int_{\mathbb{R}^+} G_k(J, J') \Psi^\pm_k(J) \Psi^\pm_k(J') \times$$

$$\Phi^\pm_k(J') dJ'. \quad (54)$$

with a symmetric kernel. Thus it is easy to show that for non-trivial solutions $\Omega$ is real. Also for $\Omega \notin \mathbb{R}$ the range of $-k \omega(J)$ it can be transformed to a Fredholm IE of the second kind. We have computed the solution of (54) for the dipole modes ($k=1$) using the equilibrium density $\Psi^\pm(J) = \frac{1}{2\pi} e^{-\frac{J^2}{\sigma^2}}$ and the behavior of $\omega$ defined in (43) is shown in Fig. 3.

then re-symmetrized by introducing $\chi^\pm_k(I) := \Phi^\pm_k(I/(1 - I)) [1 - I]^{-1}$ and then discretized in 50 equidistant nodes in $J (1 \leq i \leq 50)$. In Fig. 4 one clearly sees the continuum and the discrete eigenvalue $\Omega_0 = 0$ for the $\sigma$ mode. The $\tau$ mode spectrum in Fig. 5 shows the same type of continuum and the discrete mode $\Omega_0 = -1.513$ for the $\pi$ mode. This is in excellent agreement with our WMPT [8] and PF [9] simulations where the Yokoya factor was found to be 1.51.

7 SUMMARY AND OUTLOOK

We have derived an averaged Vlasov equation, which both simplifies and clarifies previous pioneering work. The AVE has exact equilibria which are quasi-equilibria of the exact model. This is in excellent agreement with simulations as discussed in Fig. 2. Linearization leads to $\sigma$ and $\pi$ modes in excellent agreement with simulation as shown in Figs. 4 and 5, giving further confidence in the AVE.

The next steps in this one-degree-of-freedom model are:

(1) do a more refined analysis of the integral equation of the third kind using the work of Warnock and Bart [10],

(2) study the near resonance case by generalizing the near resonance formalism in [5] and [4],

(3) include a tune split in the off resonance case,

(4) work out the error analysis.
as presented in Section 2 and (5) investigate numerics for the AVE to see if the increased step size of $O(1/\zeta)$ which is now possible can be used to advantage. We have already extended the formalism to two degrees of freedom [11] and plan to consider some form of the three degree of freedom case.

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9 REFERENCES


10 APPENDIX

In this appendix we state the basic theorems used in the text.

**Gronwall Inequality** If $\alpha$ is a real constant, $\beta(t) \geq 0$ and $\phi(t)$ are continuous real functions for $a \leq t \leq b$ which satisfy

$$\phi(t) \leq \alpha + \int_a^t \beta(s) \phi(s) \, ds , \, a \leq t \leq b ,$$  \hspace{1cm} (55)

then

$$\phi(t) \leq \alpha e^{\int_a^t \beta(s) \, ds} , \, a \leq t \leq b .$$  \hspace{1cm} (56)

**Birkhoff’s Ergodic Theorem** Let $(X, B, P)$ a probability space, $T : X \to X$ a measure preserving map and $f \in L^1(X, B, P)$. Then

1. $\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) =: \bar{f}(x)$ exists almost surely for $x \in X$ ;
2. $\bar{f}(Tx) = \bar{f}(x)$ almost surely ;
3. $\bar{f} \in L^1(X, B, P)$ and $\|\bar{f}\|_{L^1} \leq \|f\|_{L^1} ;$
4. if $A \in B$ with $T^{-1} A = A$, then $\int_A f \, dP = \int_A \bar{f} \, dP$ (this says that if $\mathcal{I}$ is the sub-$\sigma$–algebra of $B$ consisting of all the $T$–invariant sets, then $\bar{f} = E(f|\mathcal{I})$ almost surely)
5. $\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \xrightarrow{L^1} \bar{f}.$

Note that in the case when $T$ is ergodic, like in the case of $T: \mathcal{T} \to \mathcal{T}, x \mapsto (x+\mu) \mod 2\pi, \mu$ irrational, then $\mathcal{I}$ is the trivial $\sigma$–algebra and $\bar{f}$ is constant almost surely, in particular $\bar{f} = \int_X f \, dP.$