

# The Inverse Problem: Extracting Time-Like from Space-Like Data

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A practical strategy is presented and successfully implemented to determine form factors in the time-like but unphysical (below threshold) region using dispersion relations, in a model independent way without any bias towards expected resonances. Space and time-like data have been employed along with a regularization scheme to unfold and solve the integral equations. Remarkably, resonance structures with peaks for the  $\rho(770)$ ,  $\rho'(1600)$  and a structure near the  $N\bar{N}$  threshold are automatically generated. The  $\Phi$  peak is invisible thus refuting suggestions about any sizeable  $s\bar{s}$  content in the nucleon.

## 1. THE "IN PRINCIPLE" METHOD

Consider a dispersion relation (DR) for a (generic) normalized nucleon form factor (FF)  $G(s)$  with a subtraction at  $s = 0$  and  $t < 0$

$$\ln G(t) = \frac{t\sqrt{s_o-t}}{\pi} \int_{s_o}^{\infty} \frac{\ln |G(s)| ds}{s(s-t)\sqrt{s-s_o}}. \quad (1)$$

For  $s > s_o$  (the lowest mesonic threshold,  $4m_\pi^2$  for even and  $9m_\pi^2$  for odd G parity  $N\bar{N}$  channels), the phase  $\delta(s)$  of the FF defined as  $G(s) = |G(s)|e^{i\delta(s)}$  is given by the principal value integral

$$\delta(s) = -\frac{s\sqrt{s-s_o}}{\pi} \text{Pr} \int_{s_o}^{\infty} \frac{\ln |G(s')| ds'}{s'(s'-s)\sqrt{s'-s_o}}. \quad (2)$$

So in principle the method is very simple: (i) use space-like data on the left side of Eq. (1) and solve the integral equation to find  $\ln |G(s)|$  for  $s > s_o$ . (ii) Having determined the modulus  $\ln |G(s)|$ , use Eq. (2) to compute the phase  $\delta(s)$ .

The above procedure has been unsuccessful in the past as it is an "ill-posed" mathematical problem [1, 2]. The result depends on the input data in an unstable way and an impossible accuracy is needed before one arrives to a stable unique solution.

## 2. OUR STRATEGY

A successful method[3] has been developed by splitting the time-like region into 2 parts:

*Region I*: is the unknown unphysical region  $[s_o, 4m_N^2]$  for which the FF is to be determined.

*Region II*: consists of the physically accessible time-like region  $s > 4m_N^2$ , for which data exist and quite accurate asymptotic estimates are available.

With the above breakup, the unknown part of the integral equation is reduced to the (small) region I, which is amenable to a finite matrix analysis with some technical refinements described below in brief (the details of the developed procedure may be found in references [3] and [4]).

An integral equation of the first kind, linear in the unknown  $\ln |G|$ , can be derived

$$\ln G(t) - I(t) = \frac{t\sqrt{s_o-t}}{\pi} \int_{s_o}^{4m_N^2} \frac{\ln |G(s)| ds}{s(s-t)\sqrt{s-s_o}}, \quad (3)$$

where

$$I(t) = \frac{t\sqrt{s_o-t}}{\pi} \int_{4m_N^2}^{\infty} \frac{\ln |G(s)| ds}{s(s-t)\sqrt{s-s_o}}, \quad (4)$$

is a "known" quantity since it can be calculated directly from experimental data in the time-like region with some recipe to extrapolate them to very high  $t$  values.

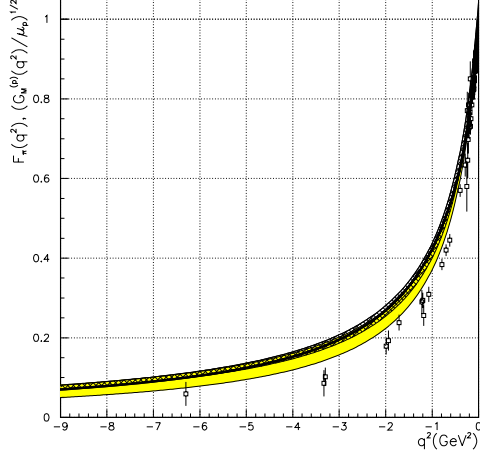


Figure 1: Pion space-like FF computed via DR without (grid band) and with (light band) subtraction, compared with pion FF space-like data and square route of proton space-like FF data fit (dark band).

To avoid instabilities usually met in solving first kind integrals such as Eq. (3), we impose a regularization scheme with smoothness:

- $I(t)$  is calculated using time-like data through a rational, smooth function with the expected asymptotic behavior. The subtraction at  $s = 0$  helps in diminishing the impact which the asymptotic behavior has on the results.
- There is a steep spike near the  $N\bar{N}$  threshold. To avoid any ensuing instabilities, the upper limit of the unphysical region has been raised to  $s_2 = 4m_N^2 + \Delta$  where  $\Delta \approx 0.5\text{GeV}^2$  and continuity is imposed there. A new DR is constructed for the region  $(4m_N^2, s_2)$  [3, 4].
- Our regularization consists in requiring the local cur-

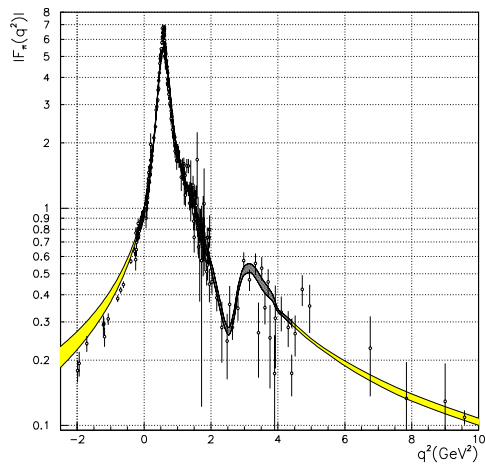


Figure 2: Pion FF computed via DR from time-like data.

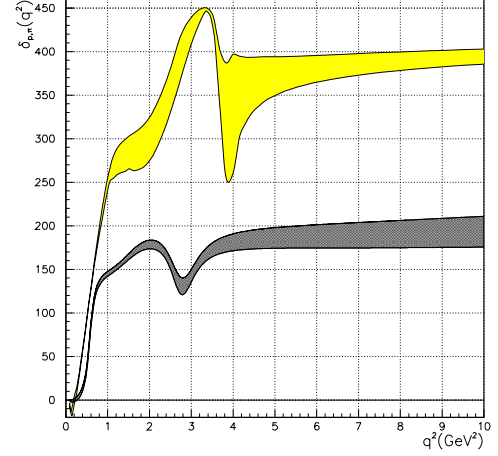


Figure 3: Phases of proton (light band) and pion (dark band) FF computed via DR.

vature of the FF in the unphysical region,  $R_2 = \int_{s_0}^{s_2} \left( \frac{d^2|G(s)|}{ds^2} \right)^2 ds$  to be limited. Instead of the second derivative of  $\ln |G|$ , as is standard[1, 2], we employ the second derivative of  $|G|$  for this purpose. The reason is that fluctuations in  $|G|$  are important only when  $|G|$  is large, while  $\ln |G|$  fluctuations would be large also when  $|G|$  is small.

- Eq. (3) is then linearized by transforming the integrals into sums over  $M = 50$  suitable sub intervals in  $s$ , with their widths increasing with  $s$ . This introduces further smoothness, by effectively integrating over any structure with a narrower half width.

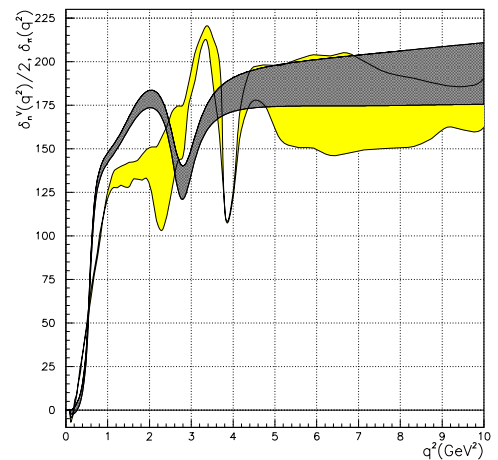


Figure 4: Phase of pion FF (dark band) compared with the half of nucleon isovector FF (light band).

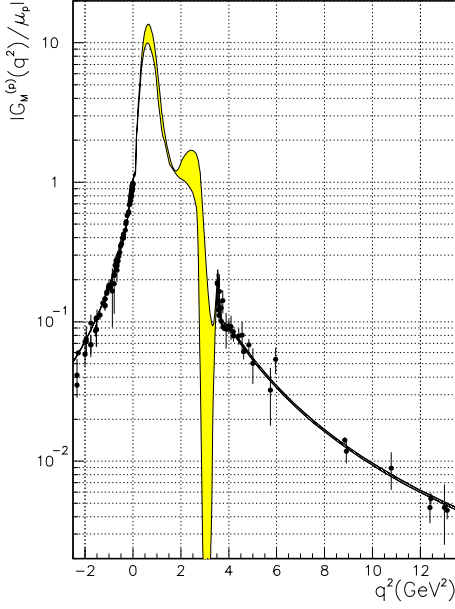


Figure 5: Proton magnetic FF computed via DR.

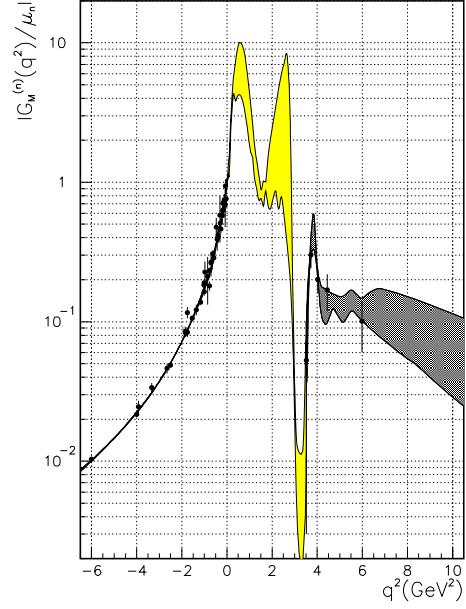


Figure 6: Neutron magnetic FF computed via DR.

- To minimize the integral

$$R_o = \sum_{i=1}^L \left[ \sum_{j=1}^M F_j \frac{t_i \sqrt{s_o - t_i}}{\pi} \int_{s_j}^{s_{j+1}} \frac{ds}{s(s-t_i)\sqrt{s-s_o}} + I(t_i) - \ln G(t_i) \right]^2, \quad (5)$$

where  $F_j = \ln |G[(s_{j+1} + s_j)/2]|$  is calculated in the middle of the  $j$ th sub interval.  $t_i$ , with  $i = 1, \dots, L$ , correspond to the experimental points available in the space-like region.

- Finally, we have

$$\mathcal{R}_{total} = R_o + \tau^6 R_2 + C.$$

The “dumping parameter”  $\tau$  has to be chosen by trial and error: if it is set too large, it will not respond to sharp structures, while unstable solutions will result if it is set too low.

- The uncertainties in the solution of Eqs. (2) and (3), due to experimental errors, were estimated by simulating new space- and time-like data according to the quoted errors and then solving the DR for each simulated set.

### 3. TEST OF THE REGULARIZATION METHOD

To test the entire procedure and also to get a suitable range for the parameter  $\tau$ , we computed the space-like pion FF using

time-like data. In the time-like region, this FF is known up to the  $J/\Psi$  mass and at higher  $Q^2$ , it was extrapolated using first order QCD [5]. In Figure (1), comparison is made with the measured (low  $Q^2$ ) space-like data. (Higher space like  $Q^2$  data points are through extrapolations from pion electroproduction data and thus may have systematic errors). We have also made other tests [3,4] obtaining good agreement with the  $\rho$  peak, the  $\rho$  width and also a dip at  $1.6 \text{ GeV}^2$  for  $\tau \approx m_\pi$  Figure (2). The phase of the pion FF approaches just above 2 GeV to its expected asymptotic value of 180 degrees Figure (3,4).

### 4. RESULTS FOR THE NUCLEON FF

Below we summarize some salient features of our findings:

- For the first time, resonant structures have been generated from “smooth” inputs Figures (5,6,7,8). The method is stable and reliable.
- The combined ( $\rho + \omega$ ) peaks and the  $\rho'(1600)$  are generated at the right mass. However, the  $\rho$  peak is much broader. Earlier analyses [6] had also found a similar discrepancy.
- No  $\Phi$  signal is visible thus signaling a very small  $s\bar{s}$  content in the nucleon.
- Phases for the nucleon are consistent with expectations:  $\delta_N \rightarrow 360^\circ$  within the error bands Figure (3).
- There is an interference pattern near threshold ( $M \approx 1.88 \text{ GeV}$ ) which may be related to baryonium Figures (5,6).

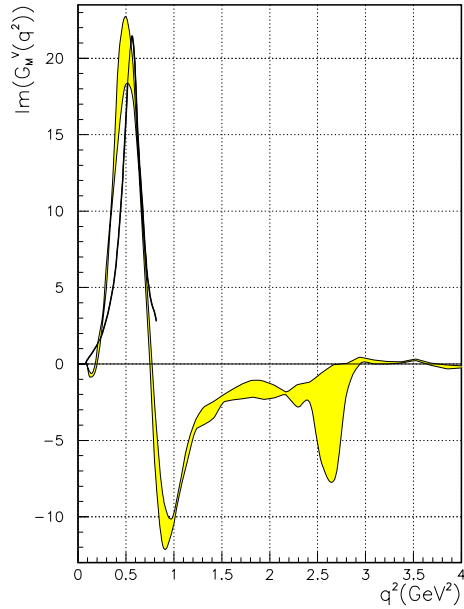


Figure 7: Imaginary part of the nucleon magnetic isovector FF computed via DR compared with expectation from unitarity relation.

- $\text{Im}G_M^{(V)}$  changes sign once and  $\text{Im}G_M^{(S)}$  appears to change sign twice and various superconvergence sum rules are all obeyed by our FF's Figures (7,8).

By way of comparison, our analysis strongly indicates that  $\text{Im}F_\pi$  does not change sign. Neglecting logarithmic factors (which we can not resolve), this would suggest (for power law behavior) that

$$|F_\pi(s)| \rightarrow |s|^{-1+\varepsilon} \text{ as } |s| \rightarrow \infty. \quad (\varepsilon > 0).$$

## 5. CONCLUSIONS

Nucleon time-like magnetic FF have been obtained in an almost model independent way by means of DR for  $\ln G(q^2)$ , using a regularization method in conjunction with space and time-like data. Resonances have been found consistent with the  $\rho(770)$  and  $\rho'(1600)$  masses. However, a very large  $\rho$  width is obtained - as in previous DR analyses. Further work is in progress to understand the sources of discrepancies as well as the relationship of our results with other DR analyses [7].

Other applications of this strategy have been discussed at this conference by Y. Srivastava (see contribution T19).

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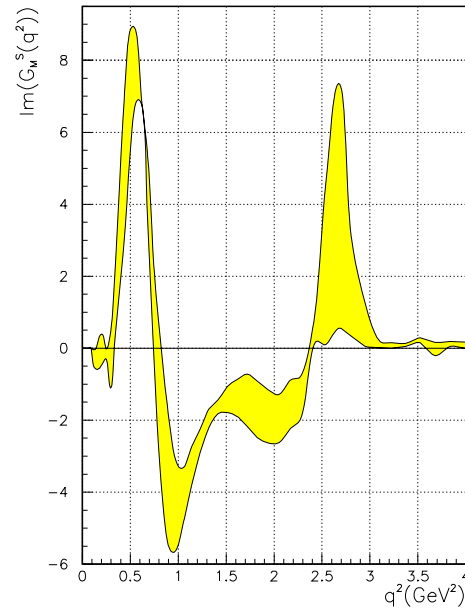


Figure 8: Imaginary part of the nucleon isoscalar FF computed via DR.

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