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QCD Sum Rule Analysis of V and A Current Correlators from τ-decay Data

Konstantin N. Zyablyuk
ITEP, Moscow

Two-point correlators of vector and axial currents, obtained from τ-decay data, are studied within the framework of perturbative QCD and Operator Product Expansion. Various sum rules, obtained from Borel transformation of the correlators in complex plane, are used to separate the contributions of different operators from each other. The analysis confirms the Q^2-dependence of the correlators in the space-like region, predicted by QCD+OPE. However the operator values are found to be in certain disagreement with the estimations, obtained from other data.

1. OBJECTIVES

Precise measurements of vector V and axial A spectral functions in hadronic τ-decays by ALEPH [1] and OPAL [2] collaborations provide us with the possibility to test various QCD aspects. Perturbation Theory (PT) and Operator Product Expansion (OPE) are the most well-established ones. Here we shall compare theoretical predictions with the data within the framework of sum rules. Particular details of this analysis can be found in [3, 4].

The 2-point correlators of charged vector and axial-vector currents

$$J = V, A: \qquad V_\mu = \bar{u}\gamma_\mu d\,, \quad A_\mu = \bar{u}\gamma_\mu\gamma_5 d$$

can be parametrized by 2 polarization functions $\Pi(q^2)$:

$$i\int e^{iqx}\left\langle J_\mu(x)J_\nu(0)^\dagger\right\rangle dx = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi_J^{(1)}(q^2) + q_\mu q_\nu \Pi_J^{(0)}(q^2). \quad (1)$$

For $q^2 = s > 0$ they have imaginary parts, the so-called spectral functions

$$v_1/a_1(s) = 2\pi\ \mathrm{Im}\ \Pi_{V/A}^{(1)}(s+i0) \quad (2)$$

which have been measured from hadronic τ-decays for $0 < s < m_\tau^2$, the plots can be found in [1, 2]. The scalar axial polarization function $\Pi_A^{(0)}$ is basically saturated by single pion decay channel. Its imaginary part a_0 is delta-function, which can be easily separated from a_1/v_1.

It turns out to be convenient to consider the sum and difference $v_1 \pm a_1$ instead of v_1 and a_1 separately. Indeed, the sum $v_1 + a_1$ is known with better accuracy, while the difference $v_1 - a_1$ does not contain perturbative terms in the massless quark limit. The QCD expressions for appropriate polarization functions can be written in the following form:

$$\Pi_V^{(1)}(s) - \Pi_A^{(1)}(s) = \sum_{k\geq 2}\frac{O_{2k}^{V-A}}{(-s)^k} \quad (3)$$

$$\Pi_V^{(1)}(s) + \Pi_A^{(1)}(s) = -\frac{1}{2\pi^2}\ln\frac{-s}{\mu^2} + \text{higher loops} + \sum_{k\geq 2}\frac{O_{2k}^{V+A}}{(-s)^k}. \quad (4)$$

The $2k$-dimensional constants $O_{2k}^{V,A}$ are the vacuum expectation values of the operators, constructed from the quark and gluon fields [5]. They have been computed up to dimension $D = 8$. The numerical values of O_{2k} cannot be determined within the perturbation theory.

Obviously the expressions (3,4) are not valid for all values of s. Exact polarization operator $\Pi(q^2)$ is known to be an analytical function of $s = q^2$ with a cut along positive real semiaxes. So it is convenient to study the QCD predictions (3,4) in the whole complex s-plane. These series are not valid for small $|s|$, where effective degrees of freedom are hadrons rather than quarks. Moreover, the higher loop perturbative terms in (4) have an unphysical cut starting from some $s = -Q_0^2 < 0$. The OPE series with a finite number of operators does not have a cut along positive real semiaxis, but has very singular behavior at $s = 0$. Based upon these speculations one may draw schematic Figure 1, displaying the region of validity of the series (3,4).

Another drawback of QCD is that these series are likely to be asymptotic, i.e. divergent for any fixed s. The way to deal with divergent series is to apply Borel transformation

$$\mathcal{B}_{M^2}\Pi = \text{pert. terms} + \sum_{k\geq 2}\frac{O_{2k}}{(k-1)!\,M^{2k}} \quad (5)$$

which improves the convergence by suppressing the higher terms. It is not clear, whether it improves the perturbative

series, which is an expansion in inverse powers of $\ln(-s)$, rather than s itself. However the expansion in $1/\ln M^2$ might be convergent.

The primary goal of the investigation is to find the numerical values of input theoretical parameters, such as $\alpha_s(m_\tau^2)$ and few operators of lowest dimensions. We shall compare the QCD result for the Borel transform (5) of the series (3,4) with the experimental values, computed by exploiting the analytical properties of exact polarization functions. In order to separate the operators from each other, we shall consider the Borel transform (5) at complex values of the argument $M^2 e^{i\phi}$. This can be alternatively understood as the Borel transformation applied to the polarization function, taken at the angle ϕ w.r.t. the real negative semiaxes in the s-plane (see Figure 1. We shall also try to find the lowest value of the Borel mass M^2, at which the comparison of QCD to the experiment can be made.

2. V – A SUM RULES

We start the analysis from the $V-A$ case (3) which is purely nonperturbative. The dispersion relation for the difference of polarization functions does not need subtractions and is written in the following way:

$$\Pi_V^{(1)}(s) - \Pi_A^{(1)}(s) = \frac{1}{2\pi^2}\int_0^\infty \frac{(v_1-a_1)(s')}{s'-s}ds' + \frac{f_\pi^2}{s}. \quad (6)$$

The last term is the kinematic pole which is specific feature of axial currents. Indeed, the r.h.s. has appropriate asymptotics: at $s \to 0$ it matches the chiral theory prediction, while the expansion at $s \to \infty$ starts from the operator of dimension $D = 4$, as it should be.

Applying the Borel transformation to (6), one gets the following sum rule:

$$\int_0^\infty e^{-s/M^2}(v_1-a_1)(s)\frac{ds}{2\pi^2} = f_\pi^2 + \sum_{k\geq 2}\frac{O_{2k}^{V-A}}{(k-1)!\,M^{2k-2}}. \quad (7)$$

One may estimate the numerical values of the operators O^{V-A} up to dimension 8 from other data:

$$O_4^{V-A} = 2(m_u+m_d)\langle\bar{q}q\rangle = -f_\pi^2 m_\pi^2$$

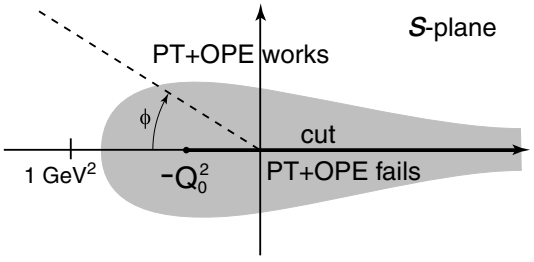

Figure 1: Region of validity of perturbation theory and operator product expansion

negligible at $s \sim 1\,\mathrm{GeV}^2$,

$$O_6^{V-A} = -\frac{64}{9}\pi\alpha_s\langle\bar{q}q\rangle^2 \approx -2\times 10^{-3}\,\mathrm{GeV}^6,$$
$$O_8^{V-A} = 8\pi\alpha_s m_0^2\langle\bar{q}q\rangle^2 \approx 2\times 10^{-3}\,\mathrm{GeV}^8, \quad (8)$$

where

$$m_0^2 = \frac{\left\langle q\hat{G}q\right\rangle}{i\langle\bar{q}q\rangle} = 0.8\pm 0.2\,\mathrm{GeV}^2 \quad (9)$$

has been found from barionic sum rules [6]. In the numerical estimation we assumed $m_u + m_d = 12\,\mathrm{MeV}$ and $\alpha_s = 0.5$ at 1 GeV^2. The factorization hypothesis was used in order to bring the operators $O_{6,8}$ to the form (8). It has internal theoretical ambiguity $\sim 1/N_c^2$ among the $D = 8$ operators, see [3].

QCD corrections to the operators O_6 have been computed in [7]. They turn out to be large and may increase the effective contribution of the $D = 6$ operator by about 50%:

$$O_6^{V-A} = -\frac{64}{9}\pi\alpha_s\langle\bar{q}q\rangle^2\left[1+\frac{\alpha_s}{\pi}\left(\frac{1}{4}\ln\frac{-s}{\mu^2}+c_6\right)\right] \approx -3\times 10^{-3}\,\mathrm{GeV}^6. \quad (10)$$

The coefficient c_6 is ambiguous: two essentially different choices were presented in [7]. In the numerical estimation (10) we used a more moderate one $c_6 = 89/48$.

One sees, that the r.h.s. of (7) has a leading term f_π^2 and relatively small (but interesting) contributions of $O_{6,8}$ at $M^2 > 0.5\,\mathrm{GeV}^2$. One way to kill f_π^2 is to differentiate (7) by M^2. This, however, inevitably increases the errors of the experimental integral. It seems more effective to perform another trick: one substitutes complex Borel mass $M^2 e^{i\phi}$ into (7) and takes imaginary part of it. The result is:

$$\int_0^\infty e^{-\frac{s}{M^2}\cos\phi}\sin\left(\frac{s}{M^2}\sin\phi\right)(v_1-a_1)(s)\frac{ds}{2\pi^2 M^2} = -\sum_{k\geq 2}\frac{\sin((k-1)\phi)}{(k-1)!}\,\frac{O_{2k}^{V-A}}{M^{2k}}. \quad (11)$$

Let us consider the angle $\phi = \pi/3$. The operator O_8 disappears from the r.h.s. of (11) and only O_6 is important in this case. The l.h.s. of (11) is shown in Figure 2a as a shaded area (the upper integration limit is m_τ^2, since there are no data beyond this point). The errors have local minimum at the point $M^2 = 0.8\,\mathrm{GeV}^2$. It happens because the $\sin(\ldots)$ in the integral has zero at $s = m_\tau^2$ and thereby suppresses large experimental errors. At this point we determine the operator O_6^{V-A} and plot the r.h.s. of (11) with this value in Figure 2a as a dashed line.

Second interesting angle is $\phi = \pi/4$. Both O_6 and O_8 contribute, but the next term with O_{10} disappears. This means that one may go to lower values of M^2 in order to reduce the experimental uncertainty. Indeed, as can be seen from Figure 2b, the agreement can be achieved down to $M^2 = 0.4\,\mathrm{GeV}^2$ in this case. At this point we obtain the most accurate value of the operator O_8.

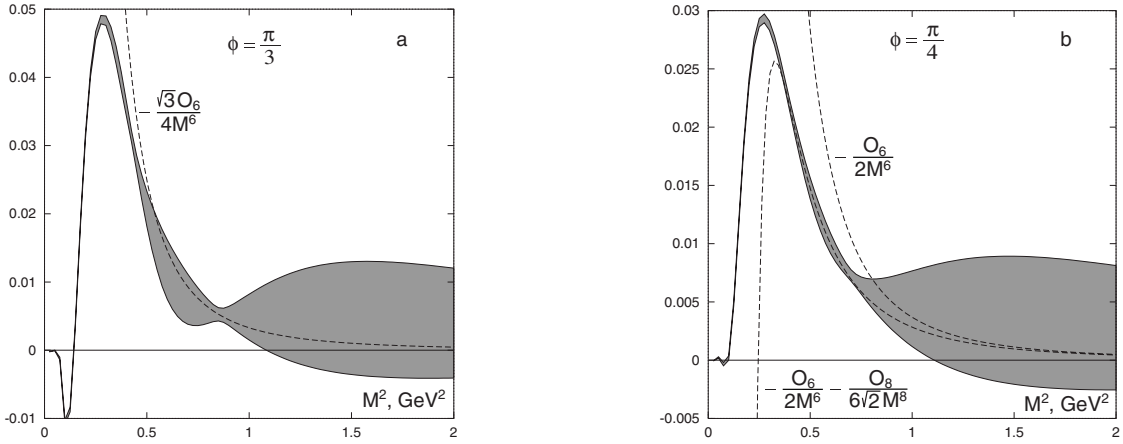

Figure 2: Sum rule (11) for $\phi = \pi/3$ (a) and $\phi = \pi/4$ (b). Dash lines display OPE prediction with operators (12).

The result of the fit:

$$\begin{aligned} O_6^{V-A} &= -(6.8 \pm 2.1) \times 10^{-3}\,\mathrm{GeV}^6 \\ O_8^{V-A} &= (7 \pm 4) \times 10^{-3}\,\mathrm{GeV}^8 \end{aligned} \qquad (12)$$

(details of the fit and error estimations are discussed in [3]). The result for O_6 is twice larger than our estimation (10). It might have different explanations: overestimated $m_u + m_d$, failure of factorization, large α_s corrections are the few ones among them. But the mass m_0^2, obtained from (12) is in agreement with (9).

3. PERTURBATIVE SERIES

Before analyzing $V + A$ sum rules, we outline the basic features of perturbative series. The QCD coupling $a \equiv \alpha_s/\pi$ is a function of the scale Q^2, determined by the renormalization group equation:

$$\frac{da}{d\ln Q^2} = -\beta(a) = -\sum_{k\geq 0} \beta_k a^{k+2} \qquad (13)$$

where the factors β_k have been computed up to 4 loops in $\overline{\mathrm{MS}}$ scheme [8]. In particular $\beta_0 = 4/9$, $\beta_1 = 4$, $\beta_2 = 10.06$ and $\beta_3 = 47.23$ for 3 flavors. The solution of the RG equation is

$$\ln \frac{Q^2}{\mu^2} = -\int_{a(\mu^2)}^{a(Q^2)} \frac{da}{\beta(a)}\,, \qquad Q^2 = -s. \qquad (14)$$

Since the integral is convergent at ∞ for any fixed order (at least with positive β_k), the coupling $a(s)$ has unphysical singularity at some negative $s = -Q_0^2$, see Figure 1. The properties of the solution of the RG equation can be understood by viewing Figure 3.

The polarization function is obtained by integrating the Adler function, which is finite and has been computed up to N^{3}LO term in $\overline{\mathrm{MS}}$ [9]:

$$\begin{aligned} D(Q^2) &= -2\pi^2 \frac{d\Pi(Q^2)}{d\ln Q^2} \\ &= 1 + a + K_2 a^2 + K_3 a^3 + \text{unknown} \end{aligned} \qquad (15)$$

where $K_2 = 1.64$ and $K_3 = 6.37$ for 3 flavors. In our calculations we shall take the theoretical uncertainty equal to the contribution of the last term in (15), $\pm K_3 a^3$. Since we do not know K_4, it is reasonable to use only a 3-loop approximation also for the β-function in (13).

The polarization function constructed in this way has unphysical cut from $s = -Q_0^2$ to $s = 0$. It is an obvious indication of QCD inapplicability at low $|s|$. However there are certain attempts to construct the perturbative functions with appropriate analytical properties on the whole s-plane, for instance by constructing an analytical QCD coupling with help of dispersion relation [10] (subtractions assumed):

$$\begin{aligned} \alpha_s(s)_{\mathrm{an}} &= \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\,\alpha_s(s')}{s'-s} ds' \\ &= \frac{\pi}{\beta_0}\left(\frac{1}{\ln(-s/\Lambda^2)} - \frac{\Lambda^2}{\Lambda^2+s}\right) + \ldots . \end{aligned} \qquad (16)$$

This way is not unique: one may write down the same dis-

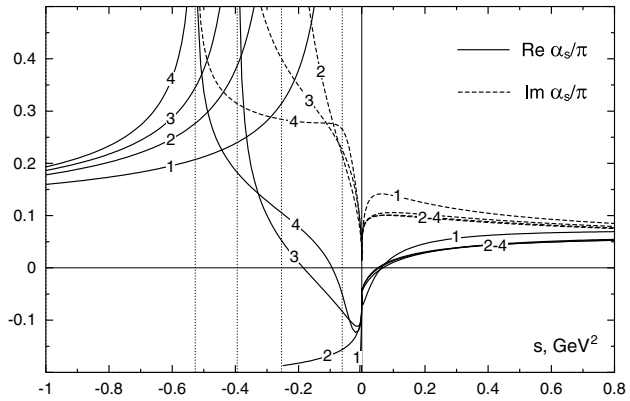

Figure 3: Real and imaginary parts of $\alpha_{\overline{\mathrm{MS}}}(s)/\pi$ as exact numerical solution of the RG equation (13) on real axes for different number of loops. The initial condition is chosen $\alpha_s = 0.355$ at $s = -m_\tau^2$. Vertical dotted lines display the position of the unphysical singularity at $s = -Q_0^2$ for each approximation ($4 \to 1$ from left to right).

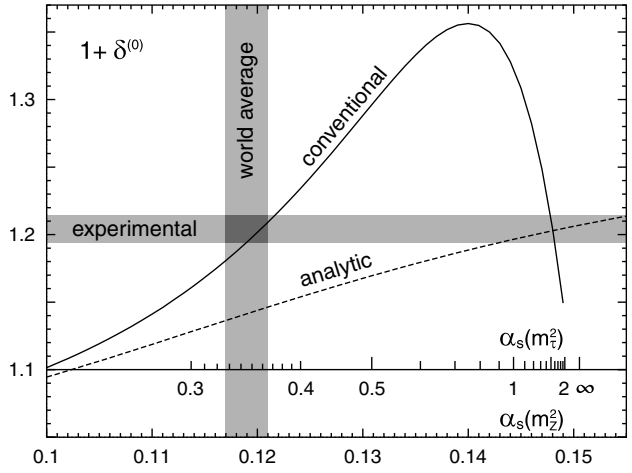

Figure 4: Correction $\delta^{(0)}$ versus $\alpha_s(m_\tau^2)$ and $\alpha_s(m_Z^2)$ in the conventional and analytic approach in the 3-loop approximation.

persion relation for the polarization function $\Pi(s)_{\rm an}$ as well. At the NLO level the result will be the same as the substitution of (16) into (15), but it is not the case for higher terms. In general, the purely logarithmic terms in analytic approach are the same as in conventional QCD, but the power terms are different, and there appears $D = 2$-like term $\sim 1/s$, absent in canonical OPE.

QCD must give correct value of the hadronic τ-decay branching ration $R_\tau \sim 3(1 + \delta^{(0)})$, which is measured with rather high accuracy. It is also weakly sensitive to the nonperturbative power corrections. The perturbative fractional correction $\delta^{(0)}$ is given by the well-known formula (for example, [11]):

$$1 + \delta^{(0)} = 1.206 \pm 0.010$$

$$= 2\pi i \oint_{|s|=m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + 2\frac{s}{m_\tau^2}\right) \Pi(s) \qquad (17)$$

$$\text{where} \qquad \Pi = \Pi_V^{(1)} + \Pi_A^{(1+0)}\,.$$

Notice, that the circle integral includes the contribution of unphysical cut, while in any analytical approach it is thrown away. The numerical results for (17) in both approaches are shown in Figure 4. It is seen that the analytical scheme predicts very large $\alpha_s(m_Z^2)$ (at $Q^2 = m_Z^2$ the difference between both approaches is not important) and, therefore, fails to agree with other data. So we shall not consider it anymore.

It follows from Figure 4:

$$\alpha_s(m_\tau^2) = 0.355 \pm 0.025. \qquad (18)$$

The error includes the theoretical uncertainty $\pm K_3 a^3$ of Adler function. This result is 3-loop, the 4-loop result with the estimation $K_4 = 25 - 50$ would give us a slightly less value within the error range. We notice also, that there is a second point on Figure 4 where the conventional curve crosses the experimental band. However it is unstable under changes of various perturbative input parameters and prescriptions and cannot be considered as a reliable one.

4. V + A SUM RULES

The operators O^{V+A} in (4) include purely gluonic condensates. In particular,

$$O_4^{V+A} = \frac{\alpha_s}{6\pi} \left\langle G^a_{\mu\nu} G^a_{\mu\nu} \right\rangle. \qquad (19)$$

The $D = 4$ gluonic condensate has been found from charmonium sum rules [5]:

$$\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle = 0.012\,\text{GeV}^4. \qquad \text{(SVZ)}$$

The $D = 6$ operator contains gluonic condensate $\sim \langle G^3 \rangle$, which is not known. The quark contribution after factorization get the form:

$$O_6^{V+A} = \frac{128}{81} \pi \alpha_s \langle \bar{q} q \rangle^2 = (1.3 \pm 0.5) \times 10^{-3}\,\text{GeV}^6. \quad (20)$$

For numerical estimation we used our $V - A$ fit (12) and added an additional error which might occur due to incomplete cancellation of two relatively large term in the sum $V + A$ after factorization. The $D = 8$ operator cannot be obtained from other data, but we estimate its upper limit as $|O_8^{V+A}| < 10^{-3}\,\text{GeV}^8$. Details given in [4]. So $O_{6,8}^{V+A}$ are essentially smaller than $O_{6,8}^{V-A}$ and perturbative terms dominate here.

Now let us define the Borel transform of the polarization function Π_{V+A}:

$$B_{\rm exp}(M^2) = \int_0^{m_\tau^2} e^{-s/M^2} (v_1 + a_1 + a_0)(s) \frac{ds}{M^2}$$

$$= B_{\rm pt}(M^2) + 2\pi^2 \sum_{k\geq 2} \frac{O_{2k}^{V+A}}{(k-1)!\, M^{2k}}. \qquad (21)$$

The perturbative part $B_{\rm pt}$ is computed numerically:

$$B_{\rm pt}(M^2) = i\pi \oint e^{-s/M^2} \Pi_{\rm pt}(s) \frac{ds}{M^2}.$$

The integration contour goes counterclockwise from $s = m_\tau^2 + i0$ to $s = m_\tau^2 - i0$ around the cut of the perturbative polarization function $\Pi_{\rm pt}$ (including the unphysical part).

Since O_8^{V+A} is small, we shall be concerned with the $D = 4, 6$ operators. They can be conveniently separated by taking the real part of the Borel transform (21) with complex argument:

$$\text{Re}\, B_{\rm exp}(M^2 e^{i\phi}) = \text{Re}\, B_{\rm pt}(M^2 e^{i\phi})$$

$$+2\pi^2 \sum_{k\geq 2} \frac{\cos(k\phi)\, O_{2k}^{V+A}}{(k-1)!\, M^{2k}}. \qquad (22)$$

At $\phi = \pi/6$ the operator O_6 disappears and at $\phi = \pi/4$ there is no O_4 in the r.h.s. Figure 5a,b displays both these possibilities. Vertical bars correspond to Re $B_{\rm exp}$, while the

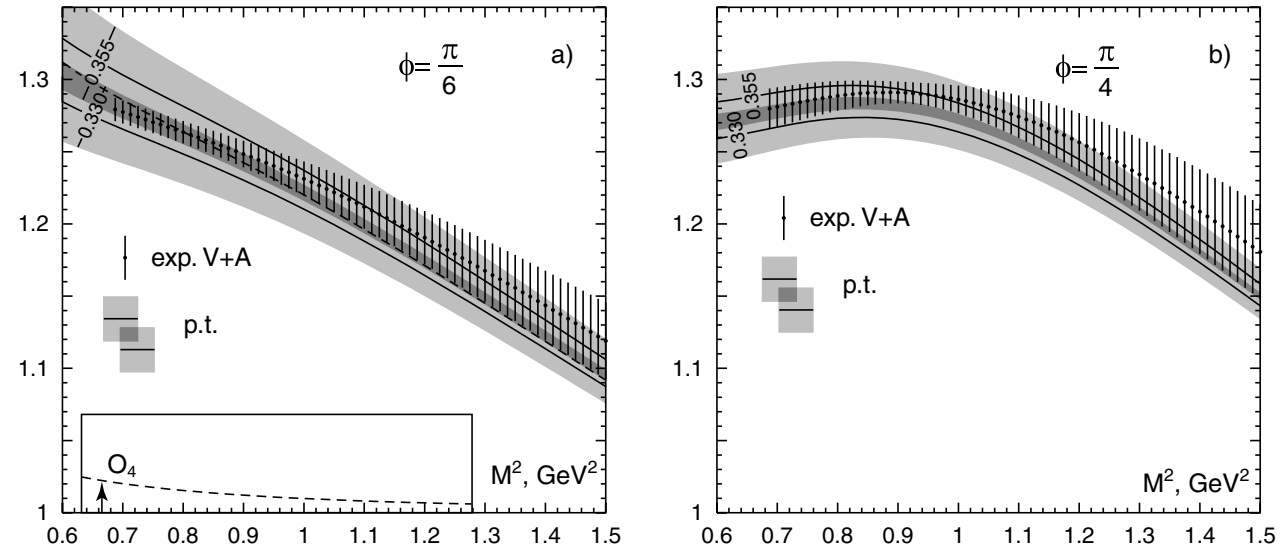

Figure 5: Sum rule (22) with $\phi = \pi/6$ (a) and $\phi = \pi/4$ (b). The dash line is the contribution of the gluonic condensate equal to the central value of (23) added to the 0.330-perturbative curve.

solid line with shaded area around it, labeled with "0.355" mark, shows purely perturbative contribution Re $B_{\rm pt}$ computed with initial condition $\alpha_s(m_\tau) = 0.355$. The same is done for $\alpha_s(m_\tau^2) = 0.330$, which is the lowest possible value within the error range (18) (for a reason which will become clear later).

Consider at first $\phi = \pi/4$. The gluonic condensate $\langle G^2 \rangle$, and consequently O_4^{V+A} must be positive. Therefore, the perturbative curve must go below the experimental one, if the discrepancy is explained by OPE. However Figure 5a shows, that the central value of 0.355-theoretical prediction goes above the experimental band for $M^2 < 0.9\,\mathrm{GeV}^2$. If we forget for a moment about theoretical uncertainty and assume that the theory should work for $M^2 > 0.6\,\mathrm{GeV}^2$, as follows from our $V - A$ analysis, this means that the condensate must be negative. It rather contradicts our expectations.

But if we take a slightly lower input $\alpha_s(m_\tau^2)$, the perturbative curve will go down. The lowest possible value is 0.330, as follows from (18). Indeed, in this case the central theoretical value is below the experimental bars. If one takes some point, say, $M^2 = 0.8\,\mathrm{GeV}^2$, then the following values of the gluonic condensate are acceptable:

$$\left\langle \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu} \right\rangle \;=\; 0.006 \pm 0.012\,\mathrm{GeV}^4$$

for

$$\alpha_s(m_\tau^2) \;=\; 0.330 \quad \text{and} \quad M^2 > 0.8\,\mathrm{GeV}^2. \quad (23)$$

Theoretical and experimental errors are added together. In principle our result (23) does not contradict the SVZ value. However, in order to achieve it, we must sit at "the very edge of errors," which seems unlikely.

Now we turn to $\phi = \pi/4$, Figure 5b. If O_6^{V+A} is positive, as OPE+factorization predict (20), its contribution to the r.h.s. of (22) must be negative. This however strongly disfavors our previous choice $\alpha_s(m_\tau^2) = 0.330$, motivated by the sign of the gluonic condensate. So it seems rather difficult to make both $\langle G^2 \rangle$ and $D = 6$ operator positive simultaneously. Here the OPE predictions are in certain disbalance with the data.

This is not however a serious disagreement. First, both O_4^{V+A} and O_6^{V+A} are small enough. Available theoretical and experimental accuracy is about 2–3%, which is not sufficient to specify the values of both operators or to say something definite about their signs.

Second, these operators are not rigorously defined objects in perturbation theory. We do not have an algorithm to find their values from the first principles. So any statement about their properties should be considered with care. Moreover, it is not clear whether one could define them independently of the perturbative series, to which they are added. In fact, different prescriptions to separate the so-called perturbative and nonperturbative terms may lead to different results, at least at the level of 1%.

5. SUMMARY

1. The $V - A$ polarization operator $\Pi^{(1)}_{V-A}(Q^2)$ is well described by OPE series for $Q^2 \geq 1\,\mathrm{GeV}^2$.

2. The operator O_6^{V-A} is approximately 2 times larger than expected from QCD and low energy theorems.

3. The $V + A$ polarization operator $\Pi^{(1+0)}_{V+A}(Q^2)$ is well described by purely perturbative terms for $Q^2 > 1\,\mathrm{GeV}^2$.

4. Current theoretical and experimental accuracy is not sufficient to determine the value of the gluonic condensate $\langle G^2 \rangle$. However it is likely to be much lower than commonly accepted SVZ value.

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REFERENCES

[1] ALEPH collaboration: R. Barate et al, Eur. J. Phys. **C4** (1998) 409
[2] OPAL collaboration: K. Ackerstaff et al, Eur. J. Phys. **C7** (1999) 571
[3] B.L. Ioffe and K.N. Zyablyuk, Nucl. Phys. **A687** (2000) 437
[4] B.V. Geshkenbein, B.L. Ioffe and K.N. Zyablyuk, hep-ph/0104048
[5] M.A. Shifman, A.I. Vainstein and V.I. Zakharov, Nucl. Phys. **B147** (1979) 385
[6] V.M. Belyaev and B.L. Ioffe, Sov.Phys. JETP **56** (1982) 493
[7] L.-E. Adam and K.G. Chetyrkin, Phys. Lett **B329** (1994) 129
[8] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, Phys. Lett. **B400** (1997) 379
[9] L.R. Surgaladze, M.A. Samuel, Phys. Rev. Lett. **66** (1990) 560, e. ibid. 2416; S.G. Gorishny, A.L. Kataev and S.A. Larin, Phys. Lett. B259 (1991) 144
[10] D.V. Shirkov and I.L. Solovtsov, Phys. Rev. Lett. **79** (1997) 1204
[11] E. Braaten, S. Narison and A. Pich, Nucl. Phys. **B373** (1992) 581