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MULTIRESOLUTION REPRESENTATIONS FOR SOLUTIONS OF VLASOV-MAXWELL-POISSON EQUATIONS

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Abstract

We present the applications of variational–wavelet approach for computing multiresolution/multiscale representation for solution of some approximations of Vlasov-Maxwell-Poisson equations.

1 INTRODUCTION

In this paper we consider the applications of a new numerical-analytical technique which is based on the methods of local nonlinear harmonic analysis or wavelet analysis to the nonlinear beam/accelerator physics problems described by some forms of Vlasov-Maxwell-Poisson equations. Such approach may be useful in all models in which it is possible and reasonable to reduce all complicated problems related with statistical distributions to the problems described by systems of nonlinear ordinary/partial differential equations with or without some (functional)constraints. Wavelet analysis is a relatively novel set of mathematical methods, which gives us the possibility to work with well-localized bases in functional spaces and gives for the general type of operators (differential, integral, pseudodifferential) in such bases the maximum sparse forms. Our approach in this paper is based on the variational-wavelet approach from [1]-[10], which allows us to consider polynomial and rational type of nonlinearities. The solution has the following multiscale/multiresolution decomposition via nonlinear high-localized eigenmodes

$$\begin{aligned} u(t,x) &= \sum_{k\in Z^2} U^k(x)V^k(t), \qquad (1)\\ V^k(t) &= V_N^{k,slow}(t) + \sum_{i\geq N} V_i^k(\omega_i^1 t), \quad \omega_i^1 \sim 2^i \\ U^k(x) &= U_M^{k,slow}(x) + \sum_{j\geq M} U_j^k(\omega_j^2 x), \quad \omega_j^2 \sim 2^j \end{aligned}$$

which corresponds to the full multiresolution expansion in all time/space scales.

Formula (1) gives us expansion into the slow part $u_{N,M}^{slow}$ and fast oscillating parts for arbitrary N, M. So, we may move from coarse scales of resolution to the finest one for obtaining more detailed information about our dynamical process. The first term in the RHS of formulae (1) corresponds on the global level of function space decomposition to resolution space and the second one to detail space. In this way we give contribution to our full solution from each scale of resolution or each time/space scale or from each nonlinear eigenmode (Fig.1). The same is correct for the contribution to power spectral density (energy spectrum): we can take into account contributions from each level/scale of resolution. Starting in part 2 from Vlasov-Maxwell-Poisson equations we consider in part 3 the approach based on variational-wavelet formulation in the bases of compactly supported wavelets or nonlinear eigenmodes.

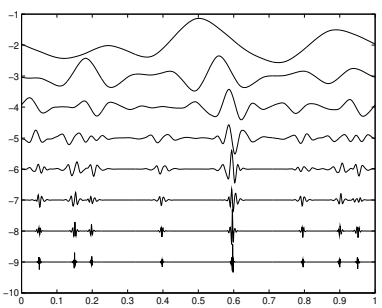

Figure 1: Multiscale/eigenmode decomposition.

2 VLASOV-MAXWELL-POISSON EQUATIONS

Analysis based on the non-linear Vlasov-Maxwell-Poisson equations leds to more clear understanding of the collective effects and nonlinear beam dynamics of high intensity beam propagation in periodic-focusing and uniform-focusing transport systems. We consider the following form of equations ([11] for setup and designation):

$$\Big\{\frac{\partial}{\partial s} + p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} - \Big[k_x(s)x + \frac{\partial \psi}{\partial x}\Big]\frac{\partial}{\partial p_x} - \Big[k_y(s)y + \frac{\partial \psi}{\partial y}\Big]\frac{\partial}{\partial p_y}\Big\} f_b(x,y,p_x,p_y,s) = 0, \qquad (2)$$

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)\psi = -\frac{2\pi K_b}{N_b}\int \mathrm{d}p_x\mathrm{d}p_y f_b, \qquad (3)$$

$$\int \mathrm{d}x\mathrm{d}y\mathrm{d}p_x\mathrm{d}p_y f_b = N_b \qquad (4)$$

The corresponding Hamiltonian for transverse single-particle motion is given by

$$H(x,y,p_x,p_y,s) = \frac{1}{2}(p_x^2+p_y^2) + \frac{1}{2}[k_x(s)x^2 + k_y(s)y^2] + H_1(x,y,p_x,p_y,s) + \psi(x,y,s), \qquad (5)$$

where H_1 is nonlinear (polynomial/rational) part of the full Hamiltonian. In case of Vlasov-Maxwell-Poisson system

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we may transform (2) into invariant form

$$\frac{\partial f_b}{\partial s} + [f, H] = 0. \tag{6}$$

3 VARIATIONAL MULTISCALE REPRESENTATION

The first main part of our consideration is some variational approach, which reduces initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second stage. Multiresolution expansion is the second main part of our construction. Because affine group of translation and dilations is inside the approach, this method resembles the action of a microscope. We have contribution to final result from each scale of resolution from the whole infinite scale of increasing closed subspaces V_j: $\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots$. The solution is parameterized by solutions of two reduced algebraical problems, one is nonlinear and the second are some linear problems, which are obtained by the method of Connection Coefficients (CC)[12]. We use compactly supported wavelet basis. Let our wavelet expansion be

$$f(x) = \sum_{\ell \in \mathbf{Z}} c_\ell \varphi_\ell(x) + \sum_{j=0}^{\infty} \sum_{k \in \mathbf{Z}} c_{jk} \psi_{jk}(x) \tag{7}$$

If $c_{jk} = 0$ for $j \geq J$, then $f(x)$ has an alternative expansion in terms of dilated scaling functions only $f(x) = \sum_{\ell \in \mathbf{Z}} c_{J\ell} \varphi_{J\ell}(x)$. This is a finite wavelet expansion, it can be written solely in terms of translated scaling functions. To solve our second associated linear problem we need to evaluate derivatives of $f(x)$ in terms of $\varphi(x)$. Let be $\varphi_\ell^n = \mathrm{d}^n \varphi_\ell(x)/\mathrm{d}x^n$. We consider computation of the wavelet - Galerkin integrals. Let $f^d(x)$ be d-derivative of function $f(x)$, then we have $f^d(x) = \sum_\ell c_l \varphi_\ell^d(x)$, and values $\varphi_\ell^d(x)$ can be expanded in terms of $\varphi(x)$

$$\begin{aligned} \varphi_\ell^d(x) &= \sum_m \lambda_m \varphi_m(x), \\ \lambda_m &= \int_{-\infty}^{\infty} \varphi_\ell^d(x) \varphi_m(x) \mathrm{d}x, \end{aligned} \tag{8}$$

where λ_m are wavelet-Galerkin integrals. The coefficients λ_m are 2-term connection coefficients. In general we need to find $(d_i \geq 0)$

$$\Lambda_{\ell_1 \ell_2 \ldots \ell_n}^{d_1 d_2 \ldots d_n} = \int_{-\infty}^{\infty} \prod \varphi_{\ell_i}^{d_i}(x) dx \tag{9}$$

For quadratic nonlinearities we need to evaluate two and three connection coefficients

$$\begin{aligned} \Lambda_\ell^{d_1 d_2} &= \int_{-\infty}^{\infty} \varphi^{d_1}(x) \varphi_\ell^{d_2}(x) dx, \\ \Lambda^{d_1 d_2 d_3} &= \int_{-\infty}^{\infty} \varphi^{d_1}(x) \varphi_\ell^{d_2}(x) \varphi_m^{d_3}(x) dx \end{aligned} \tag{10}$$

According to CC method [12] we use the next construction. When N in scaling equation is a finite even positive integer the function $\varphi(x)$ has compact support contained in $[0, N-1]$. For a fixed triple (d_1, d_2, d_3) only some $\Lambda_{\ell m}^{d_1 d_2 d_3}$ are nonzero: $2 - N \leq \ell \leq N - 2, \quad 2 - N \leq m \leq N - 2, \quad |\ell - m| \leq N - 2$. There are $M = 3N^2 - 9N + 7$ such pairs (ℓ, m). Let $\Lambda^{d_1 d_2 d_3}$ be an M-vector, whose components are numbers $\Lambda_{\ell m}^{d_1 d_2 d_3}$. Then we have the first reduced algebraical system : Λ satisfy the system of equations $(d = d_1 + d_2 + d_3)$

$$\begin{aligned} A\Lambda^{d_1 d_2 d_3} &= 2^{1-d} \Lambda^{d_1 d_2 d_3}, \\ A_{\ell,m;q,r} &= \sum_p a_p a_{q-2\ell+p} a_{r-2m+p} \end{aligned} \tag{11}$$

By moment equations we have created a system of $M + d + 1$ equations in M unknowns. It has rank M and we can obtain unique solution by combination of LU decomposition and QR algorithm. The second reduced algebraical system gives us the 2-term connection coefficients $(d = d_1 + d_2)$:

$$A\Lambda^{d_1 d_2} = 2^{1-d} \Lambda^{d_1 d_2}, \quad A_{\ell,q} = \sum_p a_p a_{q-2\ell+p} \tag{12}$$

For nonquadratic case we have analogously additional linear problems for objects (9). Solving these linear problems we obtain the coefficients of reduced nonlinear algebraical system and after that we obtain the coefficients of wavelet expansion (7). As a result we obtained the explicit time solution of our problem in the base of compactly supported wavelets. Also in our case we need to consider the extension of this approach to the case of any type of variable coefficients (periodic, regular or singular). We can produce such approach if we add in our construction additional refinement equation, which encoded all information about variable coefficients [13]. So, we need to compute only additional integrals of the form

$$\int_D b_{ij}(t) (\varphi_1)^{d_1}(2^m t - k_1) (\varphi_2)^{d_2}(2^m t - k_2) \mathrm{d}x, \tag{13}$$

where $b_{ij}(t)$ are arbitrary functions of time and trial functions φ_1, φ_2 satisfy the refinement equations:

$$\varphi_i(t) = \sum_{k \in \mathbf{Z}} a_{ik} \varphi_i(2t - k) \tag{14}$$

If we consider all computations in the class of compactly supported wavelets then only a finite number of coefficients do not vanish. To approximate the non-constant coefficients, we need choose a different refinable function φ_3 along with some local approximation scheme

$$(B_\ell f)(x) := \sum_{\alpha \in \mathbf{Z}} F_{\ell,k}(f) \varphi_3(2^\ell t - k), \tag{15}$$

where $F_{\ell,k}$ are suitable functionals supported in a small neighborhood of $2^{-\ell} k$ and then replace b_{ij} in (13) by $B_\ell b_{ij}(t)$. In particular case one can take a characteristic

function and can thus approximate non-smooth coefficients locally. To guarantee sufficient accuracy of the resulting approximation to (13) it is important to have the flexibility of choosing φ_3 different from φ_1, φ_2. In the case when D is some domain, we can write

$$b_{ij}(t) \mid_D = \sum_{0 \le k \le 2^\ell} b_{ij}(t) \chi_D(2^\ell t - k), \qquad (16)$$

where χ_D is characteristic function of D. So, if we take $\varphi_4 = \chi_D$, which is again a refinable function, then the problem of computation of (13) is reduced to the problem of calculation of integral

$$H(k_1, k_2, k_3, k_4) = H(k) = \int_{\mathbf{R}^s} \varphi_4(2^j t - k_1) \cdot \varphi_3(2^\ell t - k_2) \varphi_1^{d_1}(2^r t - k_3) \varphi_2^{d_2}(2^s t - k_4) \mathrm{d}x \quad (17)$$

The key point is that these integrals also satisfy some sort of refinement equation [13]:

$$2^{-|\mu|} H(k) = \sum_{\ell \in \mathbf{Z}} b_{2k-\ell} H(\ell), \qquad \mu = d_1 + d_2. \quad (18)$$

This equation can be interpreted as the problem of computing an eigenvector. Thus, the problem of extension of the case of variable coefficients are reduced to the same standard algebraical problem as in case of constant coefficients. So, the general scheme is the same one and we have only one more additional linear algebraic problem by which we can parameterize the solutions of corresponding problem in the same way.

So, we use wavelet bases with their good space/time localization properties to explore the dynamics of coherent structures in spatially-extended stochastic systems. After some ansatzes, reductions and constructions we give for (2)-(6) the following representation for solutions

$$u(z, s) = \sum_k \sum_\ell U_\ell^k(z) V_\ell^k(s) = \sum U_\ell^k V_\ell^k, \qquad (19)$$

where $V_\ell^k(s)$, $U_\ell^k(z)$ are both wavelets or nonlinear high-localized eigenmodes and $z = (x, y)$.

Resulting multiresolution/multiscale representation for solutions of (2)-(6) in the high-localized bases is demonstrated on Fig.2.

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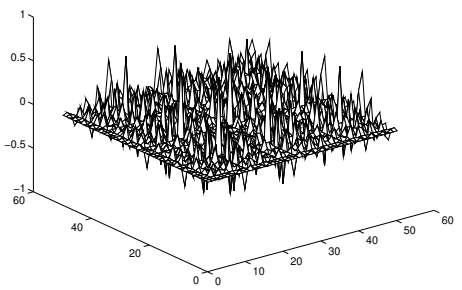

Figure 2: The solution via multiscales.

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